Financial Education - basics

basics

Jan 25, 2026

CONTENTS

This book is meant to collect some notes about financial instruments and methods for financial education, and mainly focused asset allocation.

This material is part of the **basics-books project**. It is also available as a .pdf document.

Main goal

The ultimate goal of this material is to develop an understanding of how to manage personal savings efficiently, in line with one's own reasonable objectives.

To achieve this, some intermediate goals include:

- gaining knowledge of the **macroeconomic environment**
- familiarizing with some of the most common **investment tools** (mainly bonds and stocks);
- getting used to some **common-sense** and **investing principles**: minimizing certain costs when conditions are equal, risk/reward, diversification, liquidity, and other constraints/inefficiencies
- learning **what not to do**
- and once the poor choices have been ruled out, evaluating the reasonable options for building and managing an investment portfolio, using mainly ***ETFs*** as a natural choice of a (usually) liquid asset providing diversification at low cost, even for small capitals.

Part I

Introduction

CHAPTER ONE

SUMMARY

Introduction

Financial goals; money; inflation (BC and inflation target);

Asset classes

Investing principles

Goals: …

Fundamentals: return, risk, risk-adjusted return

From 1-period to multi-period: volatility drag, composition-driven skewness,…

Principles: diversification, rebalancing,…

Rebalancing keeps the asset allocation (approximately) constant through time, preventing from single asset drift, and aiming at keeping the characteristics of the portfolio constant in time.

Diversification may improve the risk-adjusted return of the portfolio; under certain conditions, diversification may improve the return of the portfolio as well (both in terms of expected multi-period return and dispersion of the results)

Asset allocation (portfolio building and management).

- Building: MPT and CAPM, Merton's portfolio theory,…
- Management: rebalancing,…

Asset allocation

CHAPTER

TWO

REFERENCES

Here some references to othere sources, in order to reasonably organize the contents of this book

Investment and Portfolio Management - RICE - coursera - A.Ozoguz, J.Foote

Global Financial Markets

- Intro and Review of Elementary Finance Tools
- Financial system and financial assets: fixed income, equity and derivatives
- Organization of financial markets and securities trading

Portfolio Selection and Risk Management

- Intro and R/R: R/R trade-off
- Ptf construction and diversification
- Investor choices: utility functions, mean-variance preferences
- Optimal capital allocation and portfolio choice: mean-variance optimization (Modern Portfolio Theory)
- Equilibrium asset princing models: CAPM, return-beta; multi-factor models (e.g. Fama-French)

Biases and Portfolio Selection

- Efficient Market Hypotesis (EMH), and anomalies
- Biases and realistic preferences
- Inefficient markets: equity premium, volatility puzzle (?), long-run reversal to the mean, value effect, momentum
- Investor behavior

Investment Strategies and Portfolio Analysis

- Performance measurement and benchmarking
- Active vs passive investing: R^* risk-adjusted return measurements: Sharpe, Sortino, Treynor'ratio, Jensens'alpha,…;comparing rhe R^*
- Performance evaluation: style analysis and performance attribution

Capstone: Build a Winning Investment Portfolio

Using software for building ptf and assess its properties

- …

CHAPTER

THREE

1-PAGE DRAFT

3.1 Inflation and the need for investing

In contemporary economic systems, institutions targets a low non-zero inflation, usually 2%. Thus, the protection fo real valu of money is the bare minimum goal of investing.

Thus, need for investing

- starting from personal goals and constraints (saving rate, risk tolerance,$\dots$)
- knowing the available assets and their properties, e.g. 1-period — usually 1-year — return probability distribution
- knowing the fundamentals of composition of returns, and math facts about multi-period performances

First, analysis of 1-asset over 1-period, and eventually multi-asset over many periods of time.

3.2 Return, risk, and risk-adjusted return

3.2.1 Single asset, single period

The single period return of an asset X over period i from time $i-1$ and i is defined as

$$r_t := \frac{X_t - X_{t-1}}{X_{t-1}} = \frac{X_t}{X_{t-1}} - 1 \ ,$$

being X_t the value of the asset X at time t. A 1-period return of an asset X can be usually modelled as a random variable with probability distribution $p_t(r)$.

3.2.2 Single asset, multiple periods

The return compounds over many periods,

$$X_t = (1+r_t) X_{t-1} = (1+r_t) \dots (1+r_1) X_0 = \prod_{i=1}^{t} (1+r_i) X_0 = \prod_{i=1}^{t} \frac{X_i}{X_{i-1}} \ .$$

The compund average return (usually called *Compound Annual Growth Rate*, if the reference period is a year) can be defined as the constant return $r_i = r$ for all $i = 1:t$ producing the actual result, and thus

$$X_t = (1+r)^t \, X_0 \ ,$$

or

$$r = \left(\frac{X_t}{X_0} \right)^{\frac{1}{t}} - 1 \ .$$

Log-return.

$$\log \frac{X_t}{X_0} = \sum_{i=1}^{t} \log \frac{X_i}{X_{i-1}} = \sum_i \log\left(1 + r_i\right)$$
$$= t \log(1+r) \ ,$$

so that

$$\log(1+r) = \frac{1}{t} \sum_i \log\left(1 + r_i\right) \ .$$

Probability distribution, expected value and variance

Skewness of the probability density of the result $\frac{X_t}{X_0}$, typically producing *median value* lt *expected value*. Under some assumptions $(\dots)$ the expected value of the log-return has expected value that is smaller than the 1-period expected return due to dispersion (**volatility drag**)

$$\mathbb{E}[r] \sim \mathbb{E}[r_i] - \frac{\text{Var}[r_i]}{2} \ ,$$

and variance

$$\text{Var}[r] \sim \frac{\text{Var}[r_i]}{t} \ .$$

3.2.3 Multiple assets, single period

Given a portfolio with many assets of value X_{t-1} at time $t-1$, with weights

$$w_{i,t-1} := \frac{X_{i,t-1}}{X_{t-1}} = \frac{X_{i,t-1}}{\sum_i X_{i,t-1}} \ ,$$

and constant composition in time until t, the value of the final portfolio becomes

$$X_t = \sum_i X_{i,t} = \sum_i (1 + r_{i,t}) X_{i,t-1} = \sum_i r_{i,t} X_{i,t-1} + X_{t-1},$$

and the return

$$r_t := \frac{X_t}{X_{t-1}} - 1 = \frac{\sum_i r_{i,t} X_{i,t-1}}{X_{t-1}} = \sum_i r_{i,t} w_{i,t-1} = \mathbf{w}_{t-1}^T \mathbf{r}_t \ .$$

The return of the assets $\mathbf{r}$ can be modelled as a (vector) random variable. The expected value and the variance of the 1-period return of the portfolio with weights $\mathbf{w}$ read

$$\mu_r := \mathbb{E}[r] = \mathbb{E}\left[\mathbf{w}^T \mathbf{r}\right] = \mathbf{w}^T \mathbb{E}\left[\mathbf{r}\right] = \mathbf{w}^T \mu_{\mathbf{r}} \ ,$$

and

$$\begin{aligned} \sigma_r^2 := \text{Var}[r] &= \mathbb{E}\left[(r - \mathbb{E}[r])^2\right] = \\ &= \mathbb{E}\left[\mathbf{w}^T (\mathbf{r} - \mu)(\mathbf{r} - \mu)^T \mathbf{w}\right] = \\ &= \mathbf{w}^T \sigma_{\mathbf{r}}^2 \mathbf{w} \ . \end{aligned}$$

3.2.4 Multiple assets, multiple periods

Without rebalancing. No action is taken. Every individual asset of the portfolio evolves indipendently. At time 0

$$X_0 = \sum_i X_{i,0} = \sum_i w_{i,0} X_0 \ .$$

At successive times,

$$X_t = \sum_i X_{i,t} = \sum_i X_{i,0} \prod_{\tau=1}^{t} (1 + r_{i,\tau}) = X_0 \sum_i w_{i,0} \prod_{\tau=1}^{t} (1 + r_{i,\tau}) \ .$$

With rebalancing (assuming no cost…). At the end of each period, weights of individual assets are set back to the original desired value. Following this strategy, under the assumption of stationariety, the 1-period performance of the portfolio is constant, with 1-period return

$$r_1 = \mathbf{w}^T \mathbf{r} \ ,$$

and expected value and variance

$$\mu_{(1)} = \mathbf{w}^T \mu_{\mathbf{r}} \quad , \quad \sigma^2_{(1)} = \mathbf{w}^T \sigma^2_{\mathbf{r}} \mathbf{w} \ .$$

Under the assumption of "small" 1-period returns, the geometric average return reads

$$\overline{r} \sim \mathcal{N}\left(\mu_{(1)} - \frac{\sigma^2_{(1)}}{2}, \frac{\sigma^2_{(1)}}{N} \right) \ .$$

Shannon demon and the rebalancing premium: 2-asset portfolio

A 2-asset portfolio is determined by the weights $\mathbf{w} = (w_1, w_2)$ of the two assets. One-period return of the 2 assets is modeleld as a multidimensional random variable $\mathbf{r}$, whose expected value and variance read

$$\mu_{(1)} = \mathbb{E}[\mathbf{r}] = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

and

$$\sigma^2_{(1)} = \mathbb{E}[\Delta\mathbf{r}\Delta\mathbf{r}^T] = \begin{bmatrix} \sigma^2_{11} & \sigma^2_{12} \\ \sigma^2_{12} & \sigma^2_{22} \end{bmatrix} = \begin{bmatrix} \sigma^2_1 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma^2_2 \end{bmatrix} \ .$$

Under certain conditions, the expected value of the multi-period return of the diversified and rebalanced portfolio may exceed the largest expected value of multi-period returns from individual assets, i.e.

$$\mu - \frac{\sigma^2}{2} > \mu_i - \frac{\sigma^2_i}{2} \ .$$

Some algebra and analytic geometry

Let's evaluate the conditions for $i = 1$

$$\begin{aligned} 0 &< w_1\mu_1 + w_2\mu_2 - \frac{w_1^2\sigma_1^2 + 2\rho_{12}\sigma_1\sigma_2 w_1 w_2 + w_2^2\sigma_2^2}{2} - \mu_1 + \frac{\sigma_1^2}{2} = \\ &= c(w_1, w_2) \end{aligned}$$

Given the properties μ_i, σ_i, ρ_{12}, the equation $c(w_1, w_2) = 0$ is the equation of a conic section in the plane $w_1 - w_2$. With some high-school math, see general expression of conic sections, the reduced discriminant $\frac{\Delta}{4} = \frac{B^2}{4} - AC$ determines the nature of the curve. As,

$$\frac{\Delta}{4} = \frac{B^2}{4} - 4AC = \frac{\rho_{12}^2\sigma_1^2\sigma_2^2 - \sigma_1^2\sigma_2^2}{4} = \frac{1}{4}\sigma_1^2\sigma_2^2(\rho_{12}^2 - 1) \le 0 \ ,$$

the equation $c(w_1, w_2) = 0$ is the equation of an ellipse for $\rho_{12} \neq 1$, and a parabola for $\rho_{12}^2 = 1$ (perfectly correlated assets).

Non-perfectly correlated assets, $|\rho_{12}| < 1$

The general expression of the quadratic curve in the **w**-plane reads

$$0 = \frac{1}{2}\mathbf{w}^T\sigma^2\mathbf{w} - \mu^T\mathbf{w} + \mu_1^{multi}$$

In order to find the **center of the ellipse**, a translation $\mathbf{w} = \mathbf{w}_C + \mathbf{w}_1$ should produce zero 1-degree terms, i.e.

$$\begin{aligned} 0 &= \frac{1}{2}\left(\mathbf{w}_1 + \mathbf{w}_C\right)^T\sigma^2\left(\mathbf{w}_C + \mathbf{w}_1\right) - \mu^T\left(\mathbf{w}_C + \mathbf{w}_1\right) + \mu_1^{multi} = \\ &= \frac{1}{2}\mathbf{w}_1^T\sigma^2\mathbf{w}_1 + \left(2\mathbf{w}_C^T\sigma^2 - \mu^T\right)\mathbf{w}_1 + \frac{1}{2}\mathbf{w}_C^T\sigma^2\mathbf{w}_C - \mu^T\mathbf{w}_C + \mu_1^{multi} \ , \end{aligned}$$

i.e.

$$\mathbf{w}_C = \frac{1}{2}\left(\sigma^2\right)^{-1}\mu \ .$$

Using coordinates $\mathbf{x}_1$, the expression of the ellipse becomes

$$0 = \frac{1}{2}\mathbf{w}_1^T\sigma^2\mathbf{w}_1 - \mu^T\left(\sigma^2\right)^{-1}\mu + \mu_1^{multi} \ .$$

The angle of the semi-axes of the ellipse w.r.t. the coordinate axes and their lengths appears after applying the rotation transformation, $\mathbf{w}_1 = \mathbf{R}\mathbf{w}_2$, that makes the quadratic form diagonal, i.e. as the solution of the spectral decomposition of the covariance matrix,

$$\mathbf{R}^T\sigma^2\mathbf{R} =: \sigma_d^2 \ .$$

Using the results obtained for general expression of conic sections, if $\rho_1 \neq \rho_2$, the rotation angle θ satisfies

$$\tan 2\theta = \frac{2\rho_{12}\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2} \ ,$$

and the length of the semi-axes, given $\mu^T\left(\sigma^2\right)^{-1}\mu - \mu_1^{multi} > 0$[1], is

$$a_{1,2} = \sqrt{\frac{\mu^T\left(\sigma^2\right)^{-1}\mu - \mu_1^{multi}}{s_{1,2}}} \ ,$$

with the eigenvalue of the covariance matrix

$$s_{1,2} = \frac{\sigma_1^2 + \sigma_2^2}{2} \mp \frac{\sqrt{\left(\sigma_1^2 - \sigma_2^2\right)^2 + 4\rho_{12}^2\sigma_1^2\sigma_2^2}}{2}$$

[1] Otherwise the general expression doesn't represent any curve in the real plane, as that equation is never satisfied by any values of **w**.

Perfectly correlated assets, $|\rho_{12}| = 1$

The general expression of the quadratic curve in the $\mathbf{w}$-plane reads

$$0 = -c(\mathbf{w}) = \frac{1}{2}\left(w_1\sigma_1 + w_2\sigma_2\right)^2 - w_1\mu_1 - w_2\mu_2 + \mu_1 - \frac{\sigma_1^2}{2} \ .$$

At the end of any period, weights are set back to the desired fractions. At time 0

$$X_0 = \sum_i X_{i,0} = \sum_i w_i X_0 \ ,$$

At time 1 before rebalancing

$$X_1 = \sum_i X^-_{i,1} = \sum_i X_{i,0}(1 + r_{i,0}) = X_0 \left(1 + \sum_i w_i r_{i,0}\right) \ .$$

At time 1 after rebalancing

$$X_1 = \sum_i X^+_{i,1} = \sum_i w_i X_1$$

At time 2 before rebalancing

$$X_2 = \sum_i X^+_{i,1}(1 + r_{i,1}) = X_1 \sum_i w_i (1 + r_{i,1}) \ .$$

and thus

$$\frac{X_t}{X_0} = \prod_{\tau=1}^{t} \left(1 + \mathbf{w}^T \mathbf{r}_\tau\right) \ .$$

The log-return comes from

$$\begin{aligned}\log \frac{X_t}{X_0} &= \sum_{\tau=1}^{t} \log(1 + \mathbf{w}^T \mathbf{r}_\tau) = \\ &\sim \sum_{\tau=1}^{t} \left(1 + \mathbf{w}^T \mathbf{r}_\tau + \frac{1}{2}\mathbf{w}^T \mathbf{r}_\tau \mathbf{r}_\tau^T \mathbf{w}\right)\end{aligned}$$

The expected value and the variance read…

Part II

Macroeconomic Context for Investing

CHAPTER

FOUR

ACTORS

4.1 People

4.2 Private companies

4.3 Government - public

4.4 Central banks

4.5 Investment banks

4.6 Foreign regions

CHAPTER

FIVE

INFLATION

Inflation is the **rate** at which the general level of prices for goods and sevices changes.

Contents. Definition and inflation indices, with examples of indices used in Italy: NIC, FOI, IPCA; components of inflation, with details of IPCA in Italy; correlation with other macroeconomic quantities; who controls inflation; origin of inflation

5.1 Inflation Indices (e.g. in Italy)

Overall inflation is the the weighted average of inflation on different classes of goods and services, weighted for their share of expenses.

Everyone perceives its own inflation, depending on its expenses. Different indices are usually used within an economy to track inflation for some "average individual".

Different indices may differ on values of weights, and other "details" like the effect of discounts and public transfers.

As an example, three indices are used in Italy:

- **NIC** (Prezzi al Consumo per l'intera Collettività Nazionale), usually the general
- **FOI** (Prezzi al Consumo per Famiglie di Operai e Impiegati), usually used for contracts, pension and inflation-linked contracts, ex-tobacco and lotteries.
- **IPCA** (Indice Armonizzato dei Prezzi al Consumo, HIPC *Harmonized Index of Consumer Prices*), used for comparison and statistics in the EU

5.2 Weights and Price Indices of Classes of Goods and Services - Italy IPCA

National and International Institutions for Statistics (in Italy, ISTAT) provide open-access databases collecting statistics about society and economics, including data about price.

ISTAT. As an example, Italian ISTAT provides data at https://esploradati.istat.it/databrowser/#/it/dw

All the data we need here is available under the category "Prezzi" - *Prices*. In order to reach a reasonable stability of the notebook, data have been downloaded, cleaned and stored in a folder on the repository of the project.

5.2.1 Inspect Data

Before producing plots, price indices and weights of level-4 categories are visually inspected. Data are usually collected in tables.

Category Price Indices - Level-4 IPCA

Category Weights - Level-4 IPCA

5.2.2 Plots

Category weights - Level-2 IPCA

The weights assigned to IPCA (Harmonized Index of Consumer Prices) categories represent the average expenditure share of households on each category of goods and services. These weights reflect how important each category is in the consumption basket.

These weights are revised annually to account for changing consumer behavior, as one can easily realize acting on the slider of the picture below. They are the weights used in computing the overall inflation i index, as the weighted sum of inflation i_c of IPCA categories,

$$i = \sum_{c \in \text{Cat}} i_c \, w_c \, .$$

Category Prices - Level-2 IPCA

Some categories in IPCA are subject to strong seasonal effects, meaning prices follow recurring patterns during the year.

As an example:

- Clothing and Footwear: in July–August, retailers apply seasonal discounts (saldi) in Italy and prices in IPCA do include these discounts when they are actually applied in stores, as it's shown by seasonal July/August price drops
- Fresh fruits and vegetables: prone to seasonal availability, leading to fluctuating prices.
- Travel and tourism: prices rise in summer and holidays.

Seasonality can obscure underlying inflation trends: that's why **seasonally adjusted** inflation is evaluated, see below.

```
Index(['[00] Indice generale',
       '[01] -- prodotti alimentari e bevande analcoliche',
       '[02] -- bevande alcoliche e tabacchi',
       '[03] -- abbigliamento e calzature',
       '[04] -- abitazione, acqua, elettricità, gas e altri combustibili',
       '[05] -- mobili, articoli e servizi per la casa',
       '[06] -- servizi sanitari e spese per la salute', '[07] -- trasporti',
       '[08] -- comunicazioni', '[09] -- ricreazione, spettacoli e cultura',
       '[10] -- istruzione', '[11] -- servizi ricettivi e di ristorazione',
       '[12] -- altri beni e servizi'],
      dtype='object', name='Tempo')
```

Category Price Changes (Inflation) - Level-2 IPCA

The 12-month inflation rate (year-on-year or YoY) compares prices in a given month to the same month the year before. It's already less prone to seasonal effects than the month-to-month rate.

However, even YoY rates can exhibit seasonal patterns, especially in volatile components like food, energy, and clothing. In order to reduce volatility of inflation indices, it's possible to use:

- **Core inflation**, as a measure of inflation that excludes the most volatile items (e.g., unprocessed food, energy), in order to provide a smoothed measure of inflation trends.
- Statistical filtering, and moving averages

Energy post-2022

Since 2022, prices in the energy and utility sectors have shown exceptional volatility. Different causes may have contributed, like geopolitical tensions (notably, the war in Ukraine), "liberalized" electricity/gas markets in Italy where price caps were adjusted or removed. Inflation in energy and electricity was also influenced by a *base effect* (e.g., very low prices in 2020–2021 followed by spikes in 2022).

Policy interventions like tax reductions and bonuses - that are not "free" -, which may or may not be reflected in consumer prices, depending on implementation.

The use of *core inflation* in 2022–2023 was arguable, as energy prices didn't just spiked and reverted, but was/is quite a long-term shock (war, sanctions, market and supply restructuring,…); as energy price influences many other sectors, food price rose as well, due to input cost shocks /fretilizers, transports,…) not as a result of seasonality only. Using core inflation and excluding energy and food components masked the true **cost-of living** impact on households.

Category contributions to overall inflation - Level-2 IPCA

5.3 Correlations in macroeconomics with inflation

Some correlations exist[1] between inflation and other macroeconocmics quantitites.

- **Phillips Curve**: inverse relation between inflation and unemployment (in the short-run)
- **Money supply** in the long-run "*Inflation is a monetary phenomenon*", M.Friedman.

5.4 Control of Inflation

Control of inflation is one of the goals of **central banks**, like the FED and the ECB.

Central banks aims at controlling inflation, matching target inflation (usually set as 2%) by means of **monetary policy**:

- interest rates (cost of money)
- non-conventional actions, like quantitative easing (QE)/tightening (QT)

[1] …

A goverment may indirectly influence inflation with **fiscal policy**, as taxation and government spending can influence demand.

Credibility of targets, and actors through their actions and forward guidance may influence inflation as well: expectations influences inflation.

5.5 Origin of inflation

Origin of inflation?

- *short-run*, *medium-run*: cost-push, demand-pull, built-in (triangle model)
- *long-run*: "inflation is always and everywhere a monetary phenomenon" M.Friedman

CHAPTER

SIX

CHARACTERISTIC TIMES IN ECONOMY

6.1 The short run

6.2 The medium run

6.3 The long run

CHAPTER

SEVEN

POLICY

	Monetary Policy	Fiscal Policy
Controlled by	CB	Government
Main tools	IR, Money supply	Taxes, Spending, Transfers
Speed	Usually faster	Politically slower, debated
Focus	Inflation, lliquidity, credit	Employment, Income distribution
Independence		

7.1 Monetary policy

7.2 Fiscal policy

Part III

Investing Principles

CHAPTER

EIGHT

INTRODUCTION TO PRINCIPLES OF INVESTING

Investing is a core part of personal financial management—it's how individuals navigate uncertainty to meet their financial goals under real-world constraints. The most basic objective is to preserve the real value of wealth, protecting it against inflation; more ambitious goals include growing capital to fund retirement, education, or other life plans.

Sound investing requires understanding *return* and *risk* of available assets, and the fundamental *R/R trade off*. It also demands attention to **constraints** such as *liquidity* needs, *time horizon*, *acceptable volatility*, and *risk tolerance*. One of the main principle is *diversification* - which can reduce risk and, in some cases, enhance returns.

This section introduces the core concepts needed to build a robust investment strategy: how compound returns shape long-term growth, how *volatility drag* reduces expected performance, and how a clear, principle-based approaches - like *rebalancing* - may improve performance under uncertainties.

Given its set of constraints, an informed and intelligent agent, see *Portfolio construction* would take actions that try to maximise return for a given accepted risk, or minimize risk for a given desired return: this behavior can be summarized in choosing actions on a *Pareto front*, i.e. within the set of all Pareto efficient solutions.

Sections

Section	Key Concepts
0. ***Il minimo indispensabile***	
1. ***Return***	
2. ***Risk***	
3. ***Risk-Return Trade-Off***	
4. ***Diversification***	
5. ***Portfolio Construction***	
6. ***Time and Compounding***	Compounding and volatility drag
7. Disciplined Investing	PIC/PAC, rebalancing,…

CHAPTER

NINE

IL MINIMO INDISPENSABILE

Questa pagina viene scritta in lingua italiana, per alcuni semplici motivi: l'Italia ha la popolazione con il minor tasso di alfabetizzazione finanziaria tra i paesi OCSE[1], e il grado di alfabetizzazione finanziaria è legato al livello di istruzione (con correlazione positiva: in media, meno hai studiato, meno sai rispondere a domande su concetti base come inflazione, interesse composto, diversificazione del rischio, e più alta è la probabilità che tu faccia vaccate; meglio scrivere in italiano quindi se l'obiettivo è raggiungere l'italiano medio).

Prima di affrontare argomenti più complessi, è quindi meglio stabilire una linea di galleggiamento e fissare un minimo di decenza e igiene dal quale partire.

Necessità dell'investimento.

- Si vive con il valore reale dei soldi, non con quello reale
- Nel sistema attuale, gli obiettivi di inflazione sono attorno al 2%[2] — misurata con un adeguato indice dei prezzi nell'area monetaria di riferimento
- L'inflazione erode il valore reale dei soldi
- Questo rende necessario cercare dei modi per difendere il valore del risparmio/patrimonio, attraverso investimento.

Come investire: i princìpi

- rendimento, rischio, R/R: composizione rendimenti, ruolo dell'incertezza (e della volatilità), minimizzazione costi certi (a parità di altre condizioni)
- **diversificazione**
- Strumenti per investire

Come investire: costruzione di un portafogli, e gestione nel tempo. Una volta definiti gli obiettivi personali, compatibilmente con la propria tolleranza a rischi e volatilità:

- Costruzione portafogli con strumenti finanziari, secondo i princìpi di investimento
- Gestione su un orizzonte temporale medio-lungo (**ribilanciamento**,...)

[1] Paola Tamma, Why Europeans need to learn more about money, FT, e riferimenti contenuti; Mario Seminerio, tweet 2025.12.12; OECD 2022 survey.

[2] Pagine di alcune banche centrali riguardo il valore obiettivo della *politica monetaria* e le motivazioni: Banca Centrale Europea, BCE o ECB; Federal Reserve, US Fed; Bank of England, BoE; Bank of Japan; Bloomberg, China may struggle to hit inflation target even after cutting it, 2025.03.05 riguardo l'adeguamento del target della Bank of China, dal 3% al 2%. Sono numeri arbitrari? Entro una certa tolleranza probabilmente sì, anche se un numero o un intervallo va scelto, e la scelta è motivata nelle pagine delle Banche Centrali. Più importante del numero in sé, è la credibilità e la capacità delle BC di riuscire a ottenere un valore di inflazione vicino all'obiettivo, tramite la politica monetaria — anche se questa può avere dei limiti, soprattutto se non accompagnata da una politica fiscale coerente da parte dei governi: per chi fa scelte economiche risulta fondamentale che gli obiettivi di inflazione siano rispettati, e che gli attori preposti siano credibili. In un mondo normale, attese di inflazione e incertezza futura sugli obiettivi di inflazione vengono scontate dagli attori del sistema economico. Ad esempio, è lecito attendersi che un investitore che acquista un'obbligazione — statale di uno stato affidabile per escludere spread rispetto a ciò che può essere considerato "risk-free"; nella stessa valuta in cui vive per escludere rischio-cambio — a scadenza 10 anni richieda almeno un rendimento reale positivo su tale orizzonte temporale. In una situazione in cui l'investitore ha la certezza (beato lui che ha certezze) che il target di inflazione sia rispettato, può chiedere un interesse nominale netto del 2.5% se si accontenta di un rendimento reale dello 0.5% annuo. Lo stesso investitore chiederebbe lo stesso interesse se si attendesse un'inflazione superiore al target, o se non considerasse attendibile l'impegno della BC di rispettare il target, o se temesse qualche "fiammata inflazionistica"? Vedi *rischio inflazione per obbligazioni*.

9.1 Inflazione

Breve nota sull'inflazione italiana negli ultimi anni. Se necessario, usate il traduttore automatico del browser.

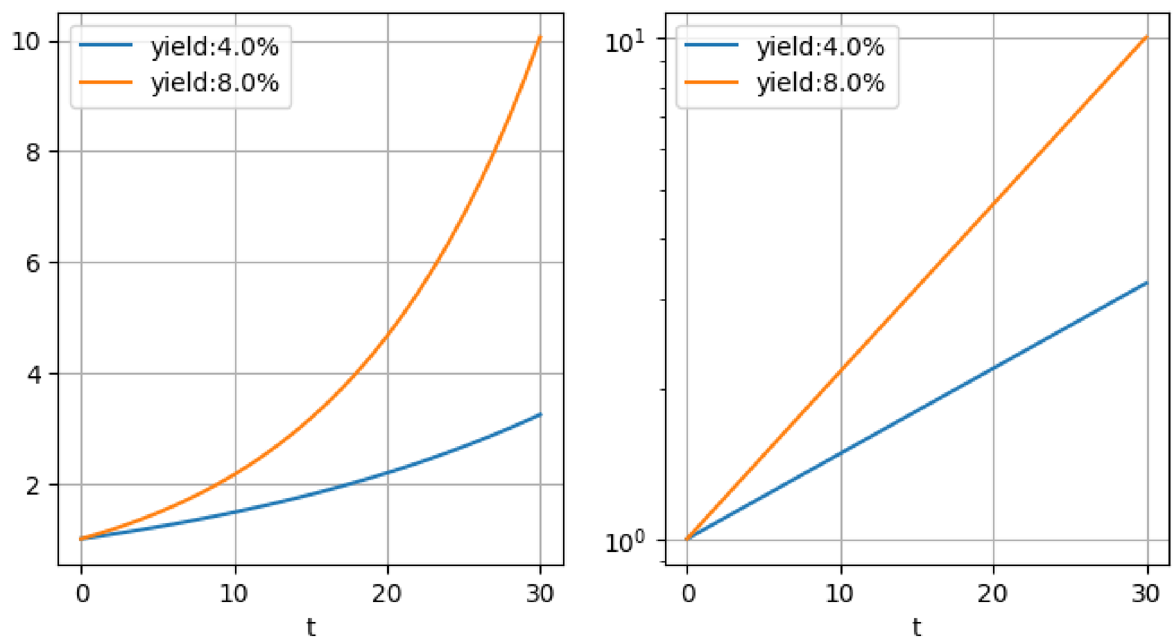
9.2 Composizione dei rendimenti/dell'interesse

9.2.1 Rendimenti costanti

Composizione del rendimento: "se migliori l'1% ogni giorno, alla fine dell'anno sei un supereroe"

- Il rendimento (o l'interesse, con reinvestimento se necessario) di un investimento su più intervalli temporale si compone con il prodotto. Una piccola differenza di rendimento sul singolo intervallo può avere effetti enormi su orizzonti temporali sufficientemente lunghi.
- Su orizzonti temporali lunghi, è misura minima di decenza e igiene usare la **scala logaritmica** nei grafici.

```
Final value after 30 periods:
1-period yield:  0.04; wealth:  3.243
1-period yield:  0.08; wealth: 10.063
```



9.2.2 Rendimenti non costanti - volatilità

- Concetti di base su composizione di rendimenti non costanti
- Volatility drag: la volatilità riduce il rendimento composto atteso

---

Se perdi il 33.33%, successivamente devi guadagnare il 50% per tornare in pari

- Se perdi il 33.33%, successivamente devi guadagnare il 50% per tornare in pari.

$$\begin{aligned} X_1 &= X_0(1-.\bar{3}) = .\bar{6}\,X_0 \\ X_2 &= X_1(1+.5) = X_0 \end{aligned}$$

- Se perdi il 50%, successivamente devi guadagnare il 100%

$$\begin{aligned} X_1 &= X_0(1-.5) = .5\,X_0 \\ X_2 &= X_1(1+1.) = X_0 \end{aligned}$$

- Se perdi il $\ell\%$ prima e guadagni il $g\%$ poi o se guadagni il $g\%$ prima e perdi il $\ell\%$ poi, alla fine il risultato è lo stesso

$$\begin{aligned} X_2^a &= X_1^a\left(1+\frac{g}{100}\right) = X_0\left(1-\frac{\ell}{100}\right)\left(1+\frac{g}{100}\right) \\ X_2^b &= X_1^b\left(1-\frac{\ell}{100}\right) = X_0\left(1+\frac{g}{100}\right)\left(1-\frac{\ell}{100}\right) \ . \end{aligned}$$

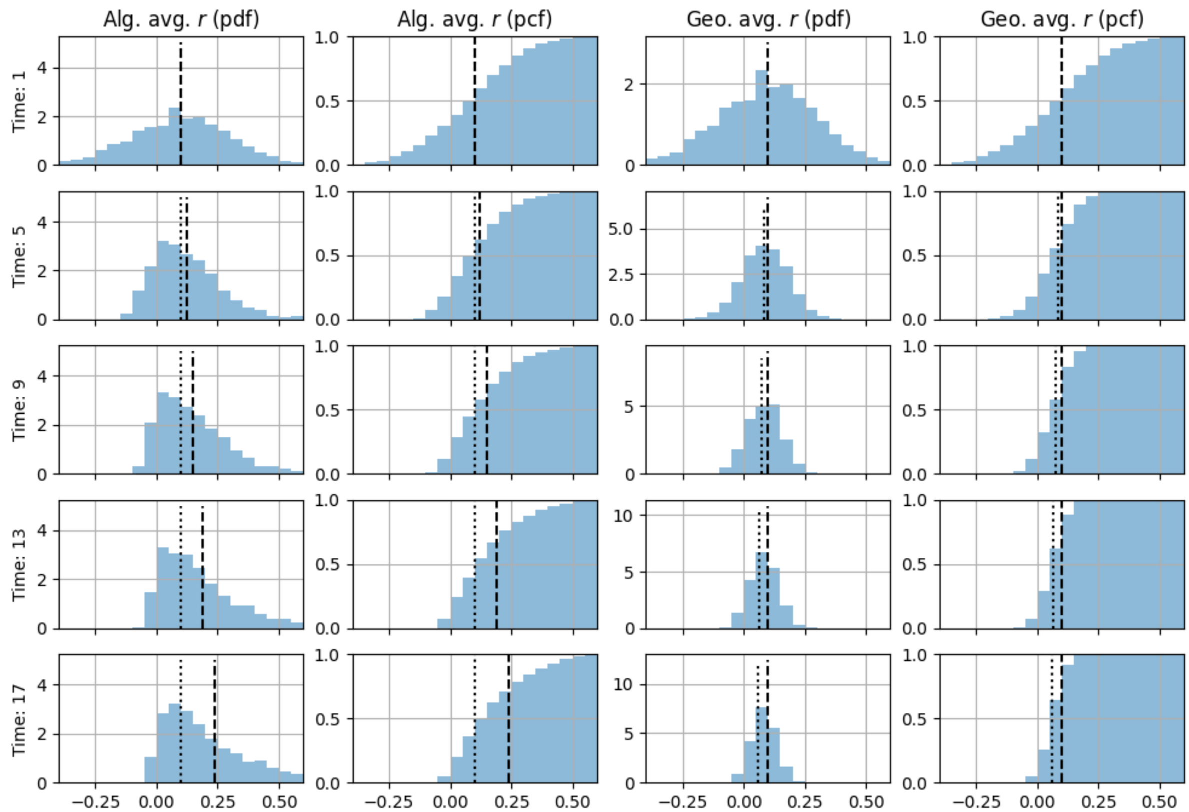
- …

Il rendimento composto produce un risultato asimmetrico

Anche un processo con rendimento di 1-periodo con probabilità simmetrica attorno a un valore medio, produce un **rendimento algebrico** medio con distribuzione di probabilità asimmetrica: in particolare si verifica la maggior parte degli eventi con risultati inferiori alla media, che risulta superiore alla mediana a causa di una minoranza di eventi estremamente positivi.

```
0.10000000000000009
0.12210200000000011
0.15088307677777799
```

```
0.18863624726100797
0.23849825205840852
```



L'esempio considera un investimento con densità di probabilità di 1-periodo con distribuzione gaussiana con valore atteso $\mu_1 = .1$ e deviazione standard $\sigma_1 = .2$

Le due righe mostrano i rendimenti algebrici (:) e geometrici (–) medi corrispondenti al rendimento atteso di un 1-periodo. La disuguaglianza $(1 + r)^N > 1 + rN$ assicura l'**effetto compounding**, che la composizione di un rendimento r su N periodi produce un risultato migliore di rendimento algebrico rN.

Così, un rendimento algebrico medio del 10% corrisponde a un rendimento composto medio decrescente nel tempo

Anni	CAGR
1	10%
5	8.4%
9	7.4%
13	6.6%
17	6.0%

Viceversa un rendimento composto medio del 10% corrisponde a un rendimento algebrico medio crescente nel tempo (effetto compounding)

Anni	Rendimento algebrico medio
1	10%
5	12.2%
9	15.1%
13	18.9%
17	23.8%

La *prima colonna* dei grafici mostra come la distribuzione di probabilità del rendimento algebrico medio diventa asimmetrica.

La *terza colonna* mostra la distribuzione del rendimento geometrico: sotto opportune ipotesi, il rendimento geometrico tende a una distribuzione gaussiana con valore atteso $\mu - \frac{\sigma^2}{2}$ (minore del rendimento di 1-periodo) e varianza $\sim \frac{\sigma}{T}$ (che quindi tende a diminuire con il tempo, come mostrato dalla distribuzione più compatta, con dispersione minore al crescere del tempo) come discusso con qualche dettaglio matematico nella sezione a proposito del *rendimento composto*.

La *seconda* e la *quarta colonna* mostrano la funzione di probabilità cumulativa del rendimento algebrico e geometrico, rispettivamente. Questi grafici mostrano come il rendimento mediano sia inferiore al rendimento atteso della distribuzione di 1-periodo: per $T = 5$, circa il 60% delle realizzazioni ha rendimento negativo, per $T = 17$, oltre il 70% delle realizzazioni ha un rendimento negativo.

```
#> Realizations
# fig, ax = plt.subplots(2,1, figsize=(5,4))
# for (i_ret, cum_ret) in enumerate(cum_rets):
#     ax[0].plot(tv, cum_ret, color=plt.cm.tab10.colors[i_ret], lw=.2)
#     ax[1].semilogy(tv, cum_ret, color=plt.cm.tab10.colors[i_ret], lw=.2)
# ax[0].grid()
# ax[1].grid()
```

Volatility-drag (I)

Una successione di eventi che comportano la perdita o il guadagno del 30%, nel tempo portano il patrimonio atteso a zero. Se la possibile perdita è del 50%, il patrimonio atteso tende a zero più velocemente.

```
#> Method 1. Succcessive independent Bernoulli = binomial distribution
from scipy.stats import binom

percs = np.array([.0, 1./30., .1, 1./3., .5, 2./3., .9, 1.-1./30., 1.])
n_percs = len(percs)
perc_pairs = [ [ i, n_percs-1-i ] for i in np.arange(int(np.floor((n_percs-1)/2))) ]

l1, b1 = binom.support(nt, p_win_1)
x = np.arange(l1,b1+1)
p = binom.cdf(x, nt, p_win_1)
y = ( 1. + gain_1 )**x * ( 1 + loss_1 )**(nt-x)

tv = np.arange(0,nt+1)
y_percs_m = []

for it in tv:
    #> Support, domain, and cdf of the Bernoulli distribution
    l1, u1 = binom.support(it, p_win_1)                # lower, upper bound
    x = np.arange(l1, u1+1)                            # domain
    p = binom.cdf(x, it, p_win_1)                      # cumulative prob
    y_1 = ( 1. + gain_1 )**x * ( 1 + loss_1 )**(u1-x)  # outcomes

    #>
    idx = np.searchsorted(p, percs, side='left')
    y_percs = y_1[idx]

    y_percs_m += [ y_percs ]

y_percs_m = np.array(y_percs_m)

fig, ax = plt.subplots(1,1, figsize=(5,3))
```

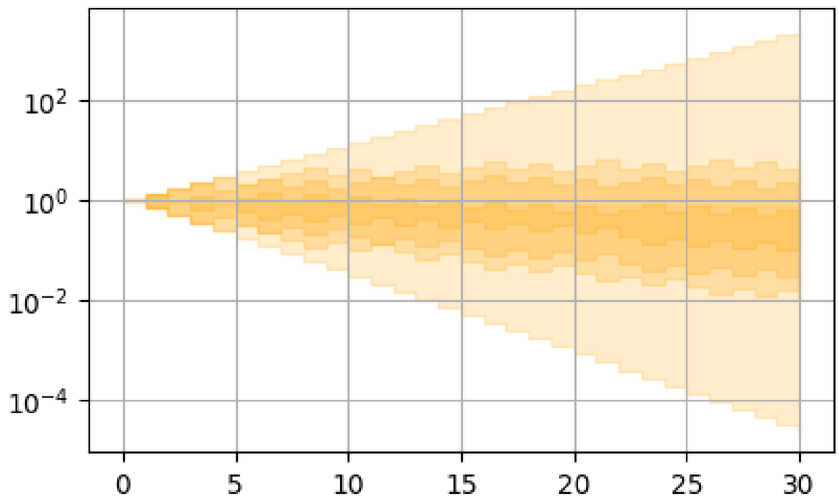
(continues on next page)

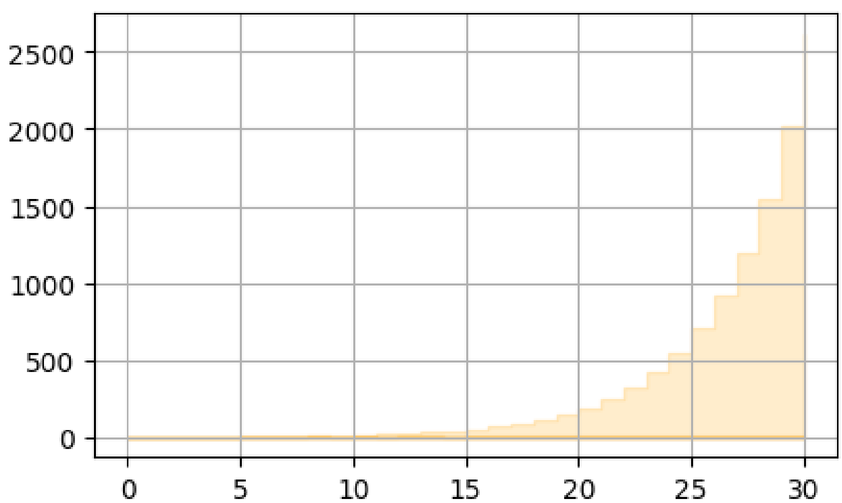
(continued from previous page)

```
# ax.step(tv, y_percs_m, where='post', color='black', lw=.5)
for perc_pair in perc_pairs:
    ax.fill_between(tv, y_percs_m[:,perc_pair[0]], y_percs_m[:,perc_pair[1]], step=
↪'post', alpha=.2, color='orange')

ax.set_yscale('log')
ax.grid()

fig, ax = plt.subplots(1,1, figsize=(5,3))
# ax.step(tv, y_percs_m, where='post', color='black', lw=.5)
for perc_pair in perc_pairs:
    ax.fill_between(tv, y_percs_m[:,perc_pair[0]], y_percs_m[:,perc_pair[1]], step=
↪'post', alpha=.2, color='orange')
ax.grid()
```






```
# #> Method 2. Realizations of the stochastic processes
# rng = np.random.default_rng().choice
# rng_params_1 = { 'a': [gain_1, loss_1], 'p': [p_win_1, p_loss_1], 'size': nt }
```

(continues on next page)

(continued from previous page)

```
# rng_params_2 = { 'a': [gain_2, loss_2], 'p': [p_win_2, p_loss_2], 'size': nt }

# w1, w2 = [], []

# for ireal in np.arange(nreals):
#     w1 += [ np.array([ 1. ] + list(np.cumprod(1+rng(**rng_params_1)))) ]
#     w2 += [ np.array([ 1. ] + list(np.cumprod(1+rng(**rng_params_2)))) ]

# w1 = np.array(w1).T
# w2 = np.array(w2).T
```

Volatility-drag (II)

A pari ritorno atteso di due investimenti, l'investimento con minore dispersione ha un rendimento composto maggiore.

Come discusso in dettaglio *più avanti*, il rendimento composto di un un'investimento con rendimento atteso μ e deviazione standard dei rendimenti σ vale $\mu - \frac{\sigma^2}{2}$.

La composizione di rendimenti simmetrici è asimmetrica

La composizione del rendimento di eventi simmetrici sul singolo intervallo temporale (es. probabilità uguale di guadagno o perdita di uno stessa quantità) porta a un rendimento asimmetrico, *con skewness*, di solito con valore mediano inferiore al valore medio: tipicamente, il valore medio risulta superiore al valore mediano a causa di eventi estremamente positivi ma estremamente rari.

9.3 Diversificazione

"Non mettere tutte le tue uova nello stesso paniere"

Citazione?

9.3.1 Portafogli con 2 asset

Dati due asset con rendimento atteso μ, e covarianza σ^2,

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$\sigma^2 = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 \end{bmatrix} = \begin{bmatrix} \sigma_{11}^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_{22}^2 \end{bmatrix}$$

un portafogli completamente investito, senza leva o short-selling, è identificato da i due pesi degli asset in protafogli

$$w_1 \ , \quad w_2 = 1 - w_1 \in [0, 1] \ .$$

Il rendimento del portafogli è la variabile casuale

$$r = w_1 r_1 + w_2 r_2 = w_1 r_1 + (1 - w_1) r_2 \ ,$$

e quindi il rendimento atteso è

$$\mu := \mathbb{E}[r] = \mathbb{E}\left[... \right] = w_1 \mu_1 + (1 - w_1) \mu_2 \ ,$$

e la varianza è

$$\begin{aligned} \sigma^2 = \mathbb{E}[(r-\mu)^T(r-\mu)] & = \mathbf{w}^T \mathbb{E}\left[(\mathbf{r} - \mu)(\mathbf{r} - \mu)^T \right] \mathbf{w} = \\ & = \mathbf{w}^T \sigma^2 \mathbf{w} = \\ & = w_1^2 \sigma_{11}^2 + 2 \rho_{12} \sigma_1 \sigma_2 w_1 w_2 + w_2^2 \sigma_{22}^2 = \\ & = w_1^2 \sigma_{11}^2 + 2 \rho_{12} \sigma_1 \sigma_2 w_1 (1 - w_1) + (1 - w_1)^2 \sigma_{22}^2 \ , \end{aligned}$$

CHAPTER

TEN

PRINCIPLES OF INVESTING

10.1 Return

Return is the reward for investing. It can come from **capital gain** (price increase of assets bought), or **periodic cashflows**, like interest (from bonds), or dividends (from stocks). Some assets produce predictable return (either nominal, or real), other assets have less predictable returns. Any asset has some level of uncertainty, or *risk*[1].

Most returns are quoted on a **per-period** basis - usually annually - and expressed as the percentage of the reward over the initial amount of the investment.

For a many-year investment, single-period returns **compound** over time.

10.1.1 Costs

While return are uncertain, at least to a certain level, usually costs - fees, expenses, taxes - or part of them, are certain. With equal other conditions, the intelligent investor should reduce costs (known), as higher costs reduce returns w/o changing the level of risk.

10.2 Risk

Risk measures uncertainty and its effects, combining probability of events and consequences of specific events. *All the assets have some systematic and some specific risks* .

Key measures (*should give info about magnitude, frequency/probability, and duration*) include:

- standard deviation or **volatility**: how much returns may deviate from their expected value),
- max loss (usually 100% can't be neglected for catastrophic although rare events), value at risk (VaR, max loss with a given probability), drawdown (maximum peak-to-trough loss)
- time-to-recover (time to recover drawdowns, in a temporal perspective)

Usually, risk metrics measure uncertainty, without discerning from positive and negative events: these metrics perceive a higher-than-expected return as a risk as well. Some metrics instead, see *Sortino ratio* in *risk-return* section, aims at quantifying only negative events as risk.

[1] Even the most safe assets could undergo some (really) **rare**, but usually (really) **catastrophic events**. Just as an example, it's hard to imagine what could happen even to bonds issued by the most (perceived and priced) safe government or institution, in case of its participation in a war.

10.3 Risk-Return Trade Off

"There's no free lunch"

Higher expected returns usually come with higher risk.

…but high risk doesn't imply high expected return

Very stupid actions usually implies poor return with high risk. Just as an example, playing Russian roulette for fun implies an expected return worse than an alternative "do-nothing and have an ice-cream instead" scenario (at least, if your goal is not to kill yourself, and your return function does not positively weight this outcome) with higher uncertainty on the final status of your health.

Sometimes the same could happen if one plays doing trading with some random meme-stocks or shit-coins.

Risk-adjusted return provides an indication of the expected return per unit of risk. Common metrics are:

- **Sharpe ratio**, comparing excess return and volatility compared with a "risk-free" asset - or a benchmark

$$S := \frac{\mathbb{E}[R - R_b]}{\sqrt{\text{var}[R - R_b]}}$$

- **Sortino ratio**

$$So := \frac{\mathbb{E}[R] - T}{\text{DR}} \ ,$$

  with T target return, and DR the downside deviation, i.e. the deviation w.r.t the target return evaluated only for returns r lower than the target return T

$$\text{DR}^2 = \int_{r=-\infty}^{T} (T - r)^2 \, f(r) \, dr \ ,$$

  being $f(r)$ the probability density function of the (continuous) random variable R representing return

10.4 Diversification

Diversification spreads risk across different investments so no single event can ruin your portfolio. Diversification works well with assets that are not - or at least they're loosely - correlated: in this case, diversification could increase return per unit of risk.

10.5 Portfolio Construction

10.6 Time

10.6.1 Compound Return

Return of multi-period investment compounds and not simply sums. Return r_n in the n^{th} period from n to $n+1$, is defined as

$$x_{n+1} = x_n (1 + r_n) \ .$$

Over N periods, the final wealth x_N can be written as a function of the initial wealth and the 1-year returns r_n,

$$x_N = x_0 \prod_{n=1}^N (1 + r_n) \ .$$

Geometric mean provides by definition the value of the average — constant — return, $\overline{r}$

$$x_N = x_0 \prod_{n=1}^N (1 + \overline{r}) = x_0 \, (1+\overline{r})^N \ \ ,$$

and thus

$$\overline{r} = \left(\frac{x_N}{x_0} \right)^{\frac{1}{N}} - 1 = \left(\prod_{n=1}^N (1 + r_n) \right)^{\frac{1}{N}} - 1 \ .$$

Logaritmic return. By the very definition of the exponential function

$$\begin{aligned} \ln \left(\overline{r} + 1 \right) & = \ln \left\{ \left(\prod_{n=1}^N (1 + r_n) \right)^{\frac{1}{N}} \right\} = \\ & = \frac{1}{N} \sum_{n=1}^N \ln \left(1 + r_n \right) \end{aligned}$$

While $\ln(1+r_n)$ is not Gaussian even if r_n is Gaussian, under the assumptions of the central limit theorem,

$$\ln \left(\overline{r} + 1 \right) \rightarrow \mathcal{N} \left(\mu_{\ln}, \frac{\sigma^2_{\ln}}{N} \right) \ \ ,$$

being $\mu_{\ln}$ and $\sigma^2_{\ln}$ the expected value and the variance of the i.i.d. random variables $\ln(1 + r_n)$. Given the probability density $p(r)$ of the 1-period return r, these values read

$$\begin{aligned} \mu_{\ln} & = \int_r p(r) \ln(1+r) \, dr \\ \sigma^2_{\ln} & = \int_r p(r) \left(\ln(1+r) - \mu_{\ln} \right)^2 \, dr \ . \end{aligned}$$

For "sufficiently small" values of $\overline{r}$, the first-order expansion $x \sim \ln(1+x)$, for $x \rightarrow 0$, provides a good approximation relating the logarithmic return and the geometric return, that thus have approximately Gaussian distribution for N "sufficiently large",

$$\overline{r} \sim \ln \left(1 + \overline{r} \right) \rightarrow \mathcal{N} \left(\mu_{\ln}, \frac{\sigma^2_{\ln}}{N} \right) \ \ .$$

Is it possible to provide an approximate value of $\mu_{\ln}$, $\sigma_{\ln}$, at least for *compact pdf* (even if not identically zero outside). Does this approximation match *GBM, Geometric Brownian motion*? See the following dropdown.

Approximated value for compact probability density

Let's broadly defined a compact probability density $p(r)$ a probability density that is *very close to* 0 outside a certain range, so small that a second-degree polynomial expansion of the function $\ln(1+r)$,

$$\ln(1+r) \sim \ln(1+\mu) + \frac{1}{1+\mu}(r - \mu) - \frac{1}{2(1+\mu)^2}(r-\mu)^2 + o\left((r-\mu)^3 \right)$$

is a good approximation of the function.

Expected value. Taking expectation of the Taylor expansion,

$$\begin{aligned}\mu_{\ln} := \mathbb{E}[\ln(1+r)] = \\ \sim \mathbb{E}\left[\ln(1+\mu) + \frac{1}{1+\mu}(r-\mu) - \frac{1}{2(1+\mu)^2}(r-\mu)^2\right] = \\ = \ln(1+\mu) - \frac{\sigma^2}{2(1+\mu)^2} \ .\end{aligned}$$

Variance. Taking the expectation of the square of the deviation from the expected value,

$$\begin{aligned}\Delta \ln(1+r) := \ln(1+r) - \mathbb{E}[\ln(1+r)] = \\ \sim \ln(1+r) - \ln(1+\mu) + \frac{\sigma^2}{2(1+\mu)^2} = \\ \sim \frac{1}{1+\mu}(r-\mu) - \frac{1}{2(1+\mu)^2}(r-\mu)^2\end{aligned}$$

gives

$$\sigma^2_{\ln} := \mathbb{E}\left[(\Delta \ln(1+r))^2\right] \sim \frac{\sigma^2}{(1+\mu)^2} \ ,$$

having neglected terms beyond the second degree.

Approximation for small values of μ. Exploiting Taylor expansion

$$f(x) \sim f(x_0) + f'(x_0)\, x + f''(x_0)\frac{x^2}{2} + \dots \ ,$$

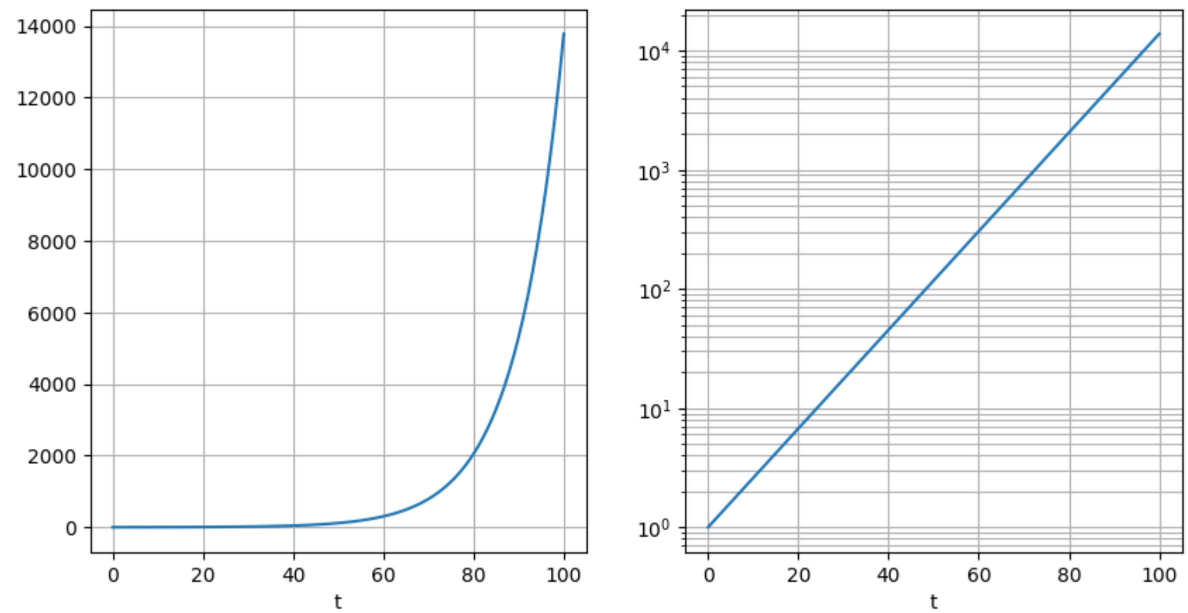
for $x_0 = 0$,

$$\begin{aligned}\ln(1+x) \sim \left[\ln(1+x_0) + \frac{1}{1+x_0}x - \frac{1}{(1+x_0)^2}\frac{x^2}{2}\right]\Bigg|_{x_0=0} = 0 + x - \frac{x^2}{2} \\ \frac{1}{(1+x)^2} \sim \left[\frac{1}{(1+x_0)^2} - \frac{2x_0}{(1+x_0)^3}x + \frac{-2+x_0}{(1+x_0)^4}\frac{x^2}{2}\right]\Bigg|_{x_0=0} = 1 - 0 \cdot x - x^2\end{aligned}$$

the low-degree approximation of the expected value and the variance of the logarithmic return reads

$$\begin{aligned}\mu_{\ln} \sim \mu - \frac{\mu^2}{2} - \frac{\sigma^2}{2}\left(1-\mu^2\right) = \mu - \frac{\sigma^2}{2} - \frac{\mu^2}{2}(1-\sigma^2) \\ \sigma^2_{\ln} \sim \sigma^2 - \sigma^2\mu^2\end{aligned}$$

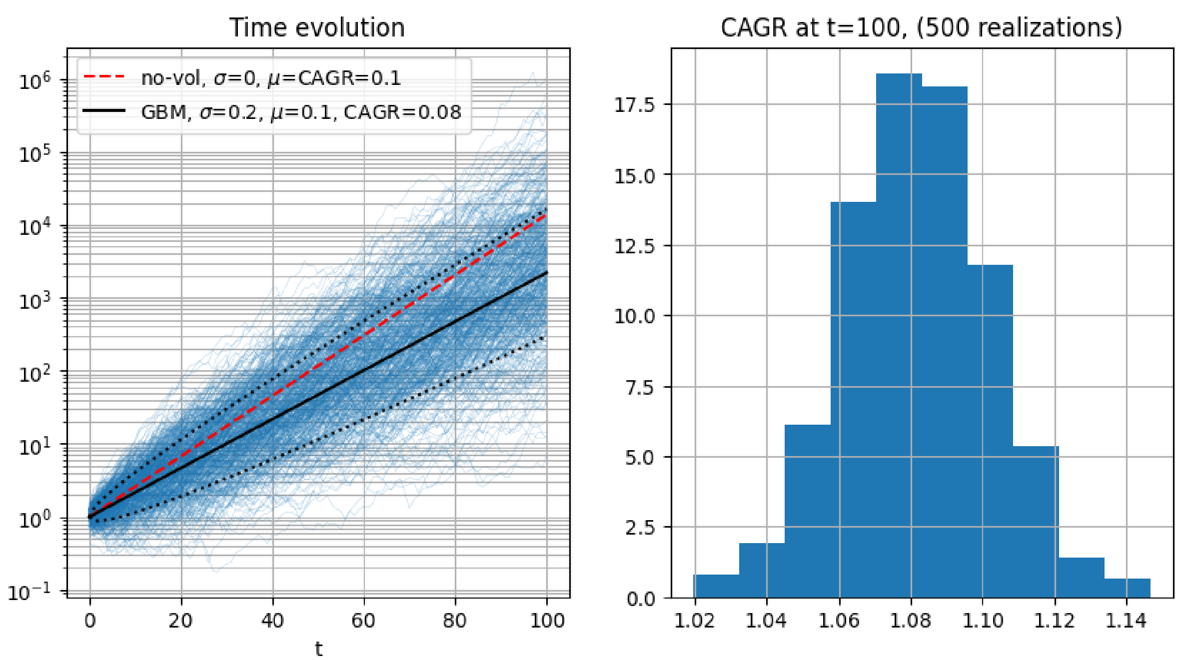
```
(Text(0.5, 0, 't'), None)
```



Volatility Drag

Under the assumption of normal distribution of 1-period return, the price of an asset with constant expected return μ and variance of returns σ can be modelled as a **geometric Brownian motion**, whose compound 1-period (usually annual, and thus the acronym *CAGR, compund annual growth rate*) growth rate has expected value

$$CAGR = \mu - \frac{\sigma^2}{2} \, .$$



todo

- *"Time and risk?" Listen to The Logic of Risk*

10.7 Disciplined Investing

10.7.1 Rebalancing

Colab Notebook, rebalancing.ipynb

Rebalancing premium

CHAPTER

ELEVEN

REBALANCING

While investing should not be confused with gambling, some games, gambling and betting strategies could provide useful toy-problems and introduction to investing principles. This is the case of gambling on a coin flip game that is discussed here and used to introduce the concept of **rebalancing**.

Rebalancing:

- reduces the dispersion of the composite return
- improves risk-adjusted return
- may also improve the expected value of the composite return as well (**rebalancing premium**, or **Shannon's demon**), for processes that meet some particular conditions

Contents

11.1 Example: coin flip game

In this section, the effect of rebalancing is discussed in a coin flip game. This game can be interpreted as a very simple model of a 2-asset portfolio, with 1 risky asset with only two outcomes (win or loss), and 1 safe asset (no *real* return).

Example 11.1.1 (Shannon demon in a coin flip game)

Starting with 100€, and a fair coin with 50% probability of for each outcome, either H: head or T: tail. If outcome is H you gain 50%, if the outcome is T you lose 33.3%.

Let's evaluate two different strategies:

1. play with all the money you have
2. at every toss, bet 50% of the amount you have

What's the expected amount at the end of the game? What's the amount distribution? …

Here the problem is investigated for a set of different strategies uniquely determined by different values of the betting fraction f.

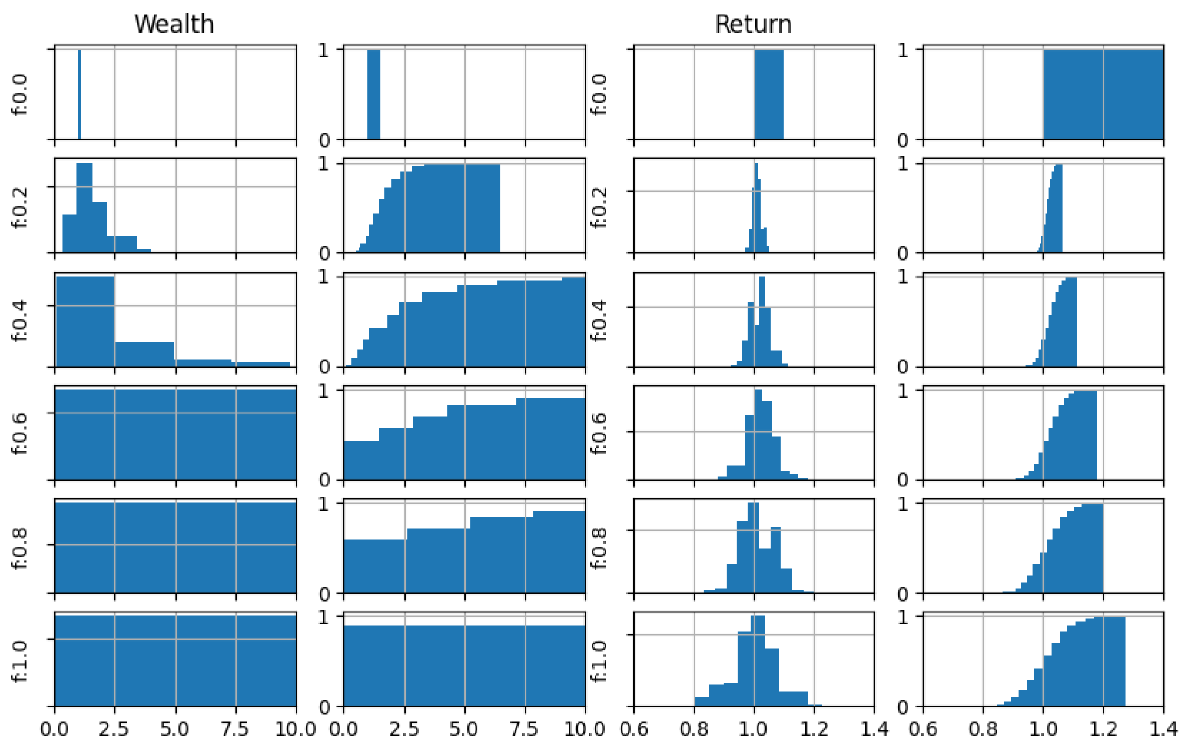
As discussed below, the *optimal fraction* (optimal in the sense of maximum expected geometric return; but does this definition of optimum meet your taste — e.g. your risk tolerance to negative results and dispersion of the results?) is

$$f^* = \frac{p}{l} - \frac{1-p}{g} = \frac{\frac{1}{2}}{\frac{1}{3}} - \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2} = .5\,.$$

```
fraction, avg wealth, avg geo return, n.negative outcomes
f:0.0, w_avg: 1.000, r_avg: 1.000, n_neg:   0
f:0.2, w_avg: 1.632, r_avg: 1.013, n_neg: 350
```

```
f:0.4, w_avg: 2.654, r_avg: 1.020, n_neg: 599
f:0.6, w_avg: 4.380, r_avg: 1.021, n_neg: 589
```

```
f:0.8, w_avg: 5.848, r_avg: 1.012, n_neg: 891
f:1.0, w_avg:12.638, r_avg: 1.004, n_neg: 832
```



todo

- Improve plot quality: very poor `hist` outcome
- Discuss:
  - difference between algebraic and geometric return (compounding)
  - probability of negative outcomes and dispersion of the results
  - probability of intermediate negative outcomes and dispersion of results (time effect and volatility)
  - comparison with no-rebalancing strategy

Example 11.1.2 (Interludio: is the coin fair?)

After tossing a coin 10 times and getting 10 Heads in a row, would you bet on Head or Tail?

Although every toss is a statistically independent event, a *very* unlikely series of outcomes under the assumption of a fair coin may induce some doubts about the truthfulness of this assumption.

See here: Inferential statistics - Hypotesis testing (Fisher) - Is the coin fair? as an example. If you want to play a bit with the notebook, just open it in Colab: the script is originally meant for $n = 30$ coin flips; just 1) set `n_flips = 10`; 2) run it again; 3) get the value of the extreme event proabability (either 10 Heads or 10 Tails) to address the question of the interludio,

i.e. the probability of getting 10 consecutive outcomes in the first 10 flips with a fair coin is less than 0.1%. A pretty low probability that the null hypothesis "the coin is fair" holds…

Here for the application of Fisher criterion for hypotesis testing for two coins with $(p_H, p_T)^1 = (0.5, 0.5)$ and $(p_H, p_T)^2 = (0.45, 0.55)$.

Example 11.1.3 (Kelly criterion for the coin flip game)

Let p the probability of a win, $q = 1 - p$ the probability of a loss, g is the net fraction gained in a win, l is the net fraction lost in a loss. The strategy is simple: always bet the fraction f of the amount you have. Is there an optimal value f^* that maximizes the expected return?

todo Clean this section containing mathematical details about Kelly's criterion.

Amount after n coin tosses

$$x_n = \ldots$$

Geometric return

$$\ln \frac{x_n}{x_0} = \ldots$$

Optimization of the expecte value of the geometric return

$$\frac{1}{n} \frac{d}{df} \mathbb{E} \left[\ln \frac{x_n}{x_0} \right] = \ldots$$

- free or constrained optimization,…

Starting with the amount x_0, the expected amount after 1 toss reads

$$\mathbb{E}\left[x_1\right] = x_0(1-f) + f x_0 (1+g) + f x_0 (1-l) \ ,$$

while the actual amount,

the amount after 1 coin toss is a random variable depending on the result of the toss, represented by a $X_1 \in \{0, 1\}$ of the coin toss, 0: loss, 1: win, reads

$$x_1 = x_0 \left(1 + fg\right)^{X_1} \left(1 - fl\right)^{1 - X_1}$$

After the second toss,

$$x_2 = x_1 \left(1 + fg\right)^{X_2} \left(1 - fl\right)^{1-X_2}$$

and after the n^{th} toss

$$x_n = x_0 \prod_{i=1}^{n} \left(1 + fg\right)^{X_i} \left(1 - fl\right)^{1-X_i}$$

The maximization of the $\ln \frac{x_n}{x_0}$ (equivalent to the maximization of $\frac{x_n}{x_0}$),

$$\ln \frac{x_n}{x_0} = \sum_{i=1}^{n} \left\{ X_i \ln\left(1 + fg\right) + \left(1 - X_i\right) \ln\left(1 - fl\right) \right\}$$

w.r.t. f gives

$$\frac{d}{df} \ln \frac{x_n}{x_0} = \sum_{i=1}^{n} \left\{ \frac{X_i\, g}{1 + fg} - \frac{(1 - X_i) l}{1 - fl} \right\}$$

The **expected value** of the logarithm of the ratio reads

$$\mathbb{E}\left[\ln \frac{x_n}{x_0}\right] = n\left\{p \ln(1 + fg) + (1 - p)\ln(1 - fl)\right\} \ ,$$

and its derivative w.r.t. f reads

$$\frac{d}{df}\mathbb{E}\left[\ln \frac{x_n}{x_0}\right] = n\left\{\frac{pg}{1 + fg} - \frac{(1 - p)l}{1 - fl}\right\}$$

and becomes zero (so that $\mathbb{E}[x_n] = x_0$ and return is zero) when

$$\begin{aligned} 0 &= pg(1 - f^*l) - (1 + f^*g)(1 - p)l = \\ &= pg - pgf^*l - l - f^*gl + pl + f^*gpl = \\ &= pg - l - f^*gl + pl \ , \end{aligned}$$

and

$$f^* = \frac{pg - l + pl}{gl} = \frac{p}{l} - \frac{1 - p}{g} \ ,$$

Is this an maximum? Compute the second-order derivative $\dfrac{d^2}{df^2}\mathbb{E}\left[\ln \frac{x_n}{x_0}\right]$, evaluate it in f^* and check if it's < 0. By direct computation, the second order derivative

$$\frac{1}{n}\frac{d^2}{df^2}\mathbb{E}\left[\ln \frac{x_n}{x_0}\right] = -\frac{pg^2}{(1 + fg)^2} - \frac{(1 - p)l^2}{(1 - fl)^2} \ ,$$

is always negative for $p \in [0, 1]$, i.e. for all the possible values of the win probability of the probability distribution. Thus, the extreme point found above is a maximum.

First order derivative for $f = 0$**.**

$$\frac{1}{n}\ \frac{d}{df}\mathbb{E}\left[\ln \frac{x_n}{x_0}\right]\Bigg|_{f=0} = pg - (1 - p)l = p(g + l) - l$$

The condition for the first derivative in $f = 0$ to be positive is equivalent to the condition of $f^* > 0$, i.e. optimal fraction is feasible, $f^* \in [0, 1]$.

Is this maximum feasible? If *no short* (betting against, $f \geq 0$) and *no leverage* (betting more than you have, $f \leq 1$) betting is possible, the fraction f must lie in range $[0, 1]$: the maximization problem is not a free optimization problem, but a simple constrained optimization problem. In these problems — under the assumptions of sufficient regularity of the function — extreme points may occur where first order derivative is zero or at the boundary of the domain.

Thus:

- either the maximum is at f^*, if $f^* \in [0,1]$, $0 \le p(l+g) - l \le lg$
- or the maximum is at $f^* = 0$: the probability is against you, so you'd better do nothing
- or the maximum is at $f^* = 1$: the probability is with you, so you'd better bet (if your **risk tolerance** accepts the **dispersion of the results**)

If *short* is allowed, there's no constraint $f \ge 0$; if *leverage* betting is allowed, there's no constraint $f \le 1$.

What's the optimal expected value? Evaluate $\mathbb{E}\left[\ln \frac{x_n}{x_0}\right](f^*)$

$$\begin{aligned}\frac{1}{n}\mathbb{E}\left[\ln \frac{x_n}{x_0}\right](f^*) &= p\ln\left(1+\frac{pg}{l}-1+p\right)+(1-p)\ln\left(1-p+\frac{(1-p)l}{g}\right) = \\ &= p\ln\left(p\left(1+\frac{g}{l}\right)\right)+(1-p)\ln\left[(1-p)\left(1+\frac{l}{g}\right)\right] = \\ &= \ln\left\{\left[p\left(1+\frac{g}{l}\right)\right]^p\left[q\left(1+\frac{l}{g}\right)\right]^q\right\}\end{aligned}$$

11.2 Resources

- *The Bull* podcast. Puntate:
  - **217.** Il modo migliore per Ribilanciare il portafoglio
  - **152.** La magia del ribilanciamento e il demone di Shannon
  - **117.** Come ribilanciare il portafoglio (e previsioni per i prossimi 10 anni)
- *R.Arnott*, over-rebalancing
- Market sentiment, Shannon's demon

CHAPTER

TWELVE

SEQUENCE RISK

12.1 Introduction

Sequence risk occurs when investment or withdrawal is distributed in time. These two scenarios may be representative of:

- Dollar Cost Averaging (**DCA**, or **PAC** in Italian for "Piano di Accumulo di Capitale")
- **Withdrawal** during old age

Sequence in time of 1-period returns may strongly influence the composite return of a portfolio.

12.1.1 Mathematical model

In a continuous-time model, sequence risk of constant-amount DCA or withdrawal can be modeled with a Goemetric Brownian Motion with "drift",

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t + C \, dt \ ,$$

being C_t the rate of investment (> 0) or withdrawal (< 0), μ, σ the expected value and variance of the rate of return. A discrete-time counterpart may be

$$\Delta X_{n,n+1} = \left(\mu_{n,n+1} + \sigma_{n,n+1} W_{n,n+1} \right) \, X_n + C_{n,n+1} \ ,$$

with $\mu_{n,n+1}$, $\sigma_{n,n+1}$ the expected value and the variance of the 1-period return, $W_{n,n+1}$ a unit-variance random variable representing the distribution of the returns from n to $n+1$, and $C_{n,n+1}$ the investment of withdrawal from n to $n+1$.

12.1.2 Constant investment or withdrawal rate: analytical solution

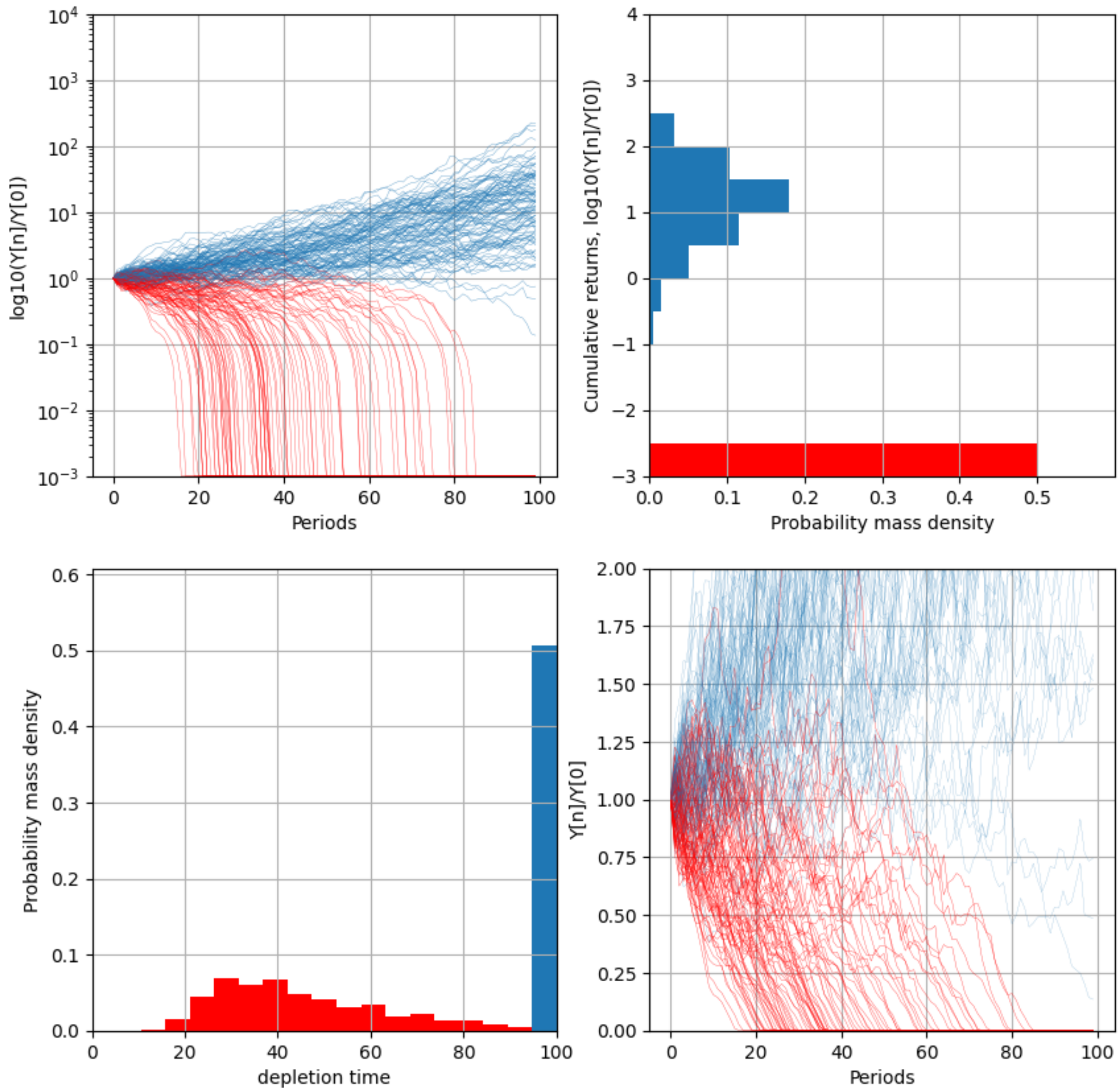
The solution of the continuous-time equation with reads

$$X_t = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t} + C \int_{s=0}^{t} e^{\left(\mu - \frac{\sigma^2}{2}\right)(t-s) + \sigma (W_t - W_s)} \, ds$$

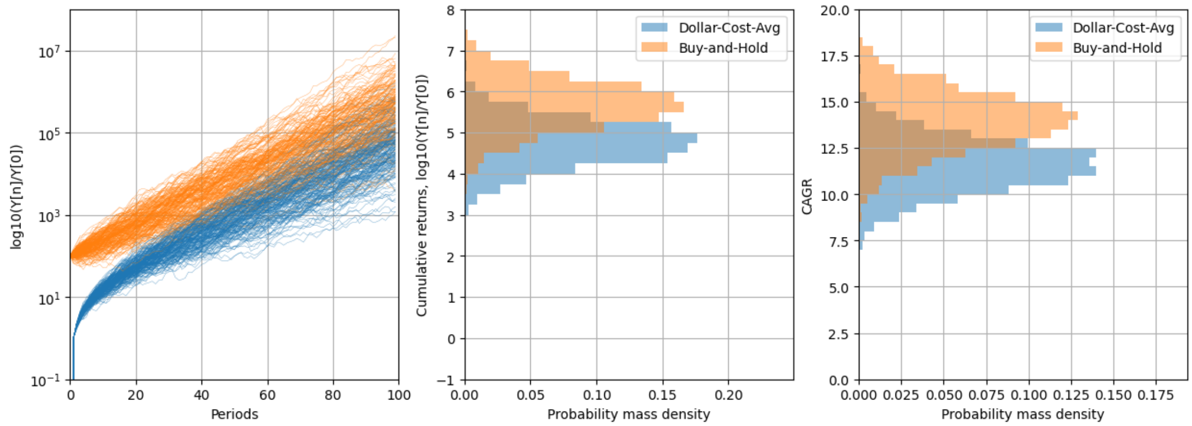
12.2 Realizations

12.2.1 Libraries and parameters

12.2.2 Constant Withdrawals



12.2.3 Dollar Cost Averaging (DCA)



Part IV

Asset classes

CHAPTER

THIRTEEN

INTRODUCTION TO ASSET CLASSES

CHAPTER

FOURTEEN

BONDS

What is a bond? ...

Characteristics of a bond. ...

Return

Risks

14.1 Return

Here the most general expression for nominal and real **yield** are derived as a function of prices, face value of coupon, taxation and year to maturity, both in case of coupon reinvestment or not (reinvestment not always possible); a closed form solution is then derived under some assumptions, like constant (or average) rates; the effect on price and yield of credit rating and rating change, coupon, year to maturity are discussed on both examples and real-world cases.

Extra:

- definition of duration
- risks: inflation; reinvesment (at lower rates) for bonds with same maturity and different coupons
- inflation linked

14.1.1 Constant coupon bonds

W/o reinvestment

At time t_0 the unit price of a bond is p_0; investing Y_0 allows to buy $N_0 = \frac{Y_0}{p_0}$ titles; each title has the right of receiving net coupon $C(1-t)$, with t taxation rate, per period (here assumed 1-year coupon range).

$$N_0 = \frac{Y_0}{p_0} = \frac{Y_0}{p_{in}} \frac{p_{in}}{p_0}$$

W/o reinvestment, the number of titles hold is constant and equal to N_0. As capital Y_i can be written as the product of unit price and number of bond in portfolio, the DCF of a bond w/o coupon reinvestment reads

$$\begin{aligned}
\widetilde{DCF} = & = -Y_0 + Y_N \prod_{k=1}^{N} (1+r_k)^{-1} + \sum_{k=1}^{N} N_0 C(1-t) \prod_{j=1}^{k} (1+r_j)^{-1} \\
& = N_0 \left[-p_0 + p_N \prod_{k=1}^{N} (1+r_k)^{-1} + C(1-t) \sum_{k=1}^{N} \prod_{j=1}^{k} (1+r_j)^{-1} \right] ,
\end{aligned}$$

This DCF must be corrected a CF at time t_N corresponding to tax on capital gain if $p_n > p_0$, discounted as

$$-N_0(p_N - p_0)\, t \prod_{k=1}^{N}(1+r_k)^{-1} \qquad \text{(only if } p_N > p_0)$$

The cumulative real return (if the discount ratio is inflation) is the ratio between the DCF and the actual value of the investment Y_0,

$$\frac{\widetilde{DCF}}{Y_0} = -1 + \frac{p_N}{p_0}\prod_{k=1}^{N}(1+r_k)^{-1} + \frac{C}{p_0}(1-t)\sum_{k=1}^{N}\prod_{j=1}^{k}(1+r_j)^{-1}$$

If the discount rate is constant, or the average (which average) discount rate is used, the expression of the cumulative return reads

$$\frac{\widetilde{DCF}}{Y_0} = -1 + \frac{p_N}{p_0}(1+r)^{-N} + \frac{C}{p_0}(1-t)\sum_{k=1}^{N}(1+r)^{-k}$$

W/ reinvestment

Time	Cashflows	ΔQuantity	Quantity	DF
0	$-Y_0$	$N_0 = \frac{Y_0}{p_0}$	$N_0 = \frac{Y_0}{p_0}$	1
1	$+N_0C(1-t)$			$(1+r_1)^{-1}$
1	$-N_0C(1-t)$	$N_1 = \frac{N_0C(1-t)}{p_1}$	$N_{0:1} = N_0 + N_1$	$(1+r_1)^{-1}$
2	$+N_{0:1}C(1-t)$			$(1+r_1)^{-1}(1+r_2)^{-1}$
2	$-N_{0:1}C(1-t)$	$N_2 = \frac{N_{0:1}C(1-t)}{p_2}$	$N_{0:2} = N_0 + N_1 + N_2$	$(1+r_1)^{-1}(1+r_2)^{-1}$
$\dots$				
$T-1$	$+N_{0:T-2}C(1-t)$			$\prod_{k=1}^{T-1}(1+r_k)^{-1}$
$T-1$	$-N_{0:T-2}C(1-t)$	$N_{T-1} = \frac{N_{0:T-2}C(1-t)}{p_{T-1}}$	$N_{0:T-1} = \sum_{k=0}^{T-1} N_k$	$\prod_{k=1}^{T-1}(1+r_k)^{-1}$
T	$+N_{0:T-1}C(1-t)$			$\prod_{k=1}^{T}(1+r_k)^{-1}$
T	$+N_{0:T-1}p_T$			$\prod_{k=1}^{T}(1+r_k)^{-1}$

All the cashflows from coupons are immediately reinvested so the DCF is

$$\begin{aligned} DCF &= -Y_0 + \underbrace{N_{0:T-1}\left(p_T + C(1-t)\right)}_{Y_T}\ \underbrace{\prod_{k=1}^{T}(1+r_k)^{-1}}_{DF_T} = \\ &= -Y_0 + Y_T\, DF_T\ , \end{aligned}$$

with

$$N_{0:T-1} = N_{0:T-2} + N_{T-1} = N_{0:T-2} + N_{0:T-2}\frac{C(1-t)}{p_{T-1}} = N_{0:T-2}\left[1+\frac{C(1-t)}{p_{T-1}}\right] =$$
$$= N_{0:T-3}\left[1+\frac{C(1-t)}{p_{T-2}}\right]\left[1+\frac{C(1-t)}{p_{T-1}}\right] =$$
$$= \cdots =$$
$$= N_{0:1}\prod_{k=2}^{T-1}\left[1+\frac{C(1-t)}{p_k}\right] =$$
$$= N_0\prod_{k=1}^{T-1}\left[1+\frac{C(1-t)}{p_k}\right] .$$

Cumulative discounted return reads

$$\frac{DCF}{Y_0} = -1 + \frac{Y_T}{Y_0}DF_T =$$
$$= -1 + \frac{N_0}{N_0\,p_0}\prod_{k=1}^{T-1}\left(1+\frac{C(1-t)}{p_k}\right)\left(p_T + C(t-1)\right)DF_T$$
$$= -1 + \frac{p_T}{p_0}\prod_{k=1}^{T}\left(1+\frac{C(1-t)}{p_k}\right)DF_T$$
$$= -1 + \frac{p_T}{p_0}\prod_{k=1}^{T}\left(\frac{1+\frac{C(1-t)}{p_k}}{1+r_k}\right) .$$

Composite discounted return is obtained, after writing the diiscounted cashflow as the difference between discounted cashflow at time t_T and t_0, $DCF = Y_T\,DF_T - T_0$,

$$(1+DCAGR)^T = \frac{Y_T\,DF_T}{Y_0} = \frac{DCF}{Y_0} + 1 = \frac{p_T}{p_0}\prod_{k=1}^{T}\left(\frac{1+\frac{C(1-t)}{p_k}}{1+r_k}\right)$$

$$DCAGR = \left(\frac{p_T}{p_0}\prod_{k=1}^{T}\frac{1+\frac{C(1-t)}{p_k}}{1+r_k}\right)^{\frac{1}{T}} - 1$$

If[1] price of the bond is constant throughout its whole life, $p_k = 1$, $\forall k = 0 : T$, and discount rate r is constant, the number of held bonds at time $T-1$ is

$$N_{0:T-1} = N_0\left(1+C(1-t)\right)^{T-1} ,$$

the discounted cashflow is

$$DCF = -N_0 + N_0\left(1+C(1-t)\right)^{T-1}\left(1+C(1-t)\right)(1+r)^{-T} =$$
$$= N_0\left[-1+\left(\frac{1+C(1-t)}{1+r}\right)^T\right] ,$$

cumulative discounted return

$$\frac{DCF}{Y_0} = -1 + \left(\frac{1+C(1-t)}{1+r}\right)^T$$

[1] It's a big if. Even if credit rating and inflation are constant throughout the life of the bond, years to maturity decreases and thus - usually - the required rate decreases as well.

and the composite discounted return reads

$$DCAGR = \frac{1 + C(1-t)}{1+r} - 1 \ .$$

14.2 Risks

- Inflation risk
- Interest rate fluctuations
- Reinvestment risk
- Rating, credit and defalut risks
- …

14.2.1 Inflation risk

Independently from any other mechanism, a change in inflation alters the real return of constant nominal yield bonds. Just as an example, if you buy a 3% nominal rate bond with expected inflation at 2%, you're aiming at 1% real return. If average inflation grows and remains constant at 4%, you get a negative -1% return.

Example 14.2.1 (Zero-coupon bond)

10-year $r = 3\%$ nominal net yield bond, with expected inflation at target inflation $i = i^* = 2\%$.

- What's the probability of getting at least the expected (and desired if you buy it?) real yield?
- What's the probability of getting negative real yield?
- What's the nominal net yield required to halve the probability of getting a real return lower than an expected 1% CAGR?

```
Final price:  100.000
Initial price to get compound return r=0.03:  74.409
```

14.3

CHAPTER

FIFTEEN

EQUITY

What's equity?

Contents

***Valuation methods*.** Comparison and intrinsic value methods.

***Financial statements*.** Introduction to financial statements of a company.

15.1 Equity Valuation

Detailed introduction

Equity valuation blends common sense, mathematics, expectations, estimation—and a bit of art. Buying shares in a company, whether directly or through a fund, means owning a (tiny) stake in a real business that produces goods and/or services and has the potential to generate earnings or free cash flows. As a shareholder, you are not just investing in market prices—you're becoming a part-owner of the enterprise. This ownership entitles you to a share of the company's profits through dividends or capital appreciation. It also comes with certain rights and responsibilities, especially during difficult periods.

When companies face financial stress or pursue growth opportunities, they may issue new shares to raise capital. This can lead to dilution, reducing the percentage ownership of existing shareholders. However, shareholders often have preemptive rights, allowing them to participate in new issuances to maintain their ownership stake. Moreover, owning equity means having a claim on the residual value of the company—what's left after all debts are paid—in both prosperous and challenging times. Understanding these dynamics is crucial to valuing equity: you're not just buying into today's performance, but into a stream of future cash flows and the complex, evolving structure of ownership.

Sensitivity analysis could provide an estimate of the effects of different parameters/assumptions on the final result.

Different valuation methods exist, and can be broadly classified in

- **comparison** approach: P/E, EV/EBITDA, or other indices used to compare companies of the same sector, marked, dimension,…[1]
- **intrinsic value** approach, based on **DCF**
- …other methods for general firms (cost approach,…); valuation of financials;…

[1] It's not always possible to find "equivalent" companies for the comparison…; P/E, EV/EBITDA,… whould be projected into the future to keep into account future in the value of a firm.

15.1.1 Comparison

15.1.2 Intrinsic value

- Future cash flows are estimated,
- CFs are discounted, usually for the $WACC$ (Weighted Average Cost of Capital) to find the NPV (net present value) of the **enterprise value** EV
- Cash and equivalents are added to the NPV to find the **equity value**

$WACC$

$$WACC = \frac{E}{V}R_e + \frac{D}{V}R_d(1-t)$$

being R_e the **cost of equity** and R_d the **cost of debt** (maybe the easiest part to estimated accurately, since the debt structure is usually known/programmed). The factor $(1-t)$ usually appears as interest payments are tax-deductible.

Equity Risk Premium R_e - Sharpe

Following W.Sharpe, equity risk premium can be estimated as

$$R_e = R_f + (R_m + R_f)\beta \ ,$$

being R_f the risk-free rate (usally 10Y US Treasuries), and R_m the annual return of the market/sector of the investment, β is a measure of risk or stock volatility of returns of the investment relative to that of the market/sector.

15.2 Three Financial Statements

Financial statements are written records that illustrates the business activities and the financial performance of a company. In most cases they are audited to ensure accuracy for tax, financing, or investing purposes.

Uses. *Management* uses them for decision-making, budgeting and performance evaluation. *Investors* use them to asses profitability, financial health, future performance, and creditworthiness (especially *lenders*).

- **Income statement**: company performance (profit and loss) over a period. Broadly speaking:

  $$\text{net earnings} = (\text{revenues} - \text{total expenses}) \times (1 - \text{tax rate}),$$

  with total expenses = operative (labor + non-labor + DA) + Interest (due to debt holders), and "partial earnings" EBITDA, EBIT, EBT with trivial definition (Earnings Before: I:interest, DA: depreciation and amortization, T: tax)
- **Balance sheet**: financial position at a specific point in time, in terms of:
  - assets: cash and equivalent + acc.receiv. + inventory + PPE (Plant property and equipment, subject to CapEx and depreciation, $PPE(n) = PPE(n-1) + \text{CapEx}(n) + \text{DA}(n)$
  - liabilities: debt + acc.pay.

- equity:

$$\text{retained earnings}(n) = \text{retained earnings}(n-1) + \text{net earnings}(n) - \text{dividends}(n)$$
$$\text{shareholder equity}(n) = \text{equity capital}(n) + \text{retained earnings}(n)\ ,$$

  being retained earnings(n) the **cumulative** retained earnings not distributed to shareholders.

  The 2 contributions shareholder equity, total liabilities shows how the compnay's asset are financed: either through capital raised or retained earnings (equity), or through debt (liabilities). The **identity**

$$\text{total liabilities} + \text{shareholders equity} = \text{total asset}$$

  must hold in a proper filled balance.

- **Cash flow statement** tracks the flows of cash in and out of the business over a period. Cashflows over a period modifies cash,

$$\text{closing cash}(n) = \text{opening cash}(n) + \text{total cashflow}(n)$$
$$\text{opening cash}(n) = \text{closing cash}(n-1)$$

  Cashflows are usually classified as:

  - operating CF (DA is added back to net income, since it's not a cashflow going anywhere; it lowers income, but it's not a cashflow)
  - investing CF
  - financing CF

$$\text{Op.CF}(n) = \text{net earnings}(n) + \text{DA}(n) - \Delta\text{WC}(n)$$
$$\text{Inv.CF}(n) = \text{investment in PPE}(n)$$
$$\text{Fin.CF}(n) = \text{issuance of debt}(n) + \text{issuance of equity}(n) - \text{dividends}(n)$$

  being $\text{WC}(n) = \text{acc.rec}(n) + \text{inventory}(n) - \text{acc.pay}(n)$ the **working capital**.

CHAPTER

SIXTEEN

ETFS

Part V

Asset allocation and Investing

CHAPTER

SEVENTEEN

INTRODUCTION TO INVESTING

CHAPTER EIGHTEEN

MODERN PORTFOLIO THEORY AND CAPITAL ASSET PRICING MODEL

Assets

Expected value μ and covariance σ^2 of expected returns of single assets are defined.

18.1 Modern Portfolio Theory

Given a set of N assets with expected return $\mu_i = \{\mu\}_i$, $i = 1 : N$, and covariance (matrix) σ^2, a portfolio is defined by the weights (percentage) $w_i = \{\mathbf{w}\}_i$ of the individual assets in the portfolio.

Return of the portfolio. The return of an amount x_0 invested in a portfolio with asset allocation w_i is found evaluating the amount x_1 after the period for which the return is defined,

$$x_1 = \sum_{i=1}^{N} x_0 w_i \left(1 + r_i\right) \ .$$

Not-fully invested portfolio can be represented with an asset 0 (cash with zero return). The return is defined as

$$r = \frac{x_1 - x_0}{x_0} = \frac{x_1}{x_0} - 1 = \sum_{i=1}^{N} w_i (1 + r_i) - 1 \ .$$

If (or as? Even with not-fully invested portfolio, or portfolio w/leverage) $\sum_{i=1}^{N} w_i = 1$, the return of the portfolio reads

$$r = \sum_{i=1}^{N} w_i r_i = \mathbf{w}^T \mathbf{r} \ .$$

Expected value and variance of the return of the portfolio. The expected value and the variance of the return thus read

$$\mu = \mathbb{E}[r] = \mathbb{E}[\mathbf{w}^T \mathbf{r}] = \mathbf{w}^T \mu \ .$$

and

$$\begin{aligned}
\sigma^2 & = \mathbb{E}[(r - \mu)^2] = \\
& = \mathbb{E}\left[((\mathbf{r} - \mu)^T \mathbf{w})^T ((\mathbf{r} - \mu)^T \mathbf{w}) \right] = \\
& = \mathbf{w}^T \mathbb{E}\left[(\mathbf{r} - \mu)(\mathbf{r} - \mu)^T \right] \mathbf{w} = \mathbf{w}^T \sigma^2 \mathbf{w} \ .
\end{aligned}$$

Optimal portfolio. The evaluation of weights $\mathbf{w}^*$ of optimal asset allocation for MPT is recast as the optimization problem of *finding asset allocation with minimum volatility (variance) for the given expected return $\overline{\mu}$, under some constraints about asset allocation*

$$\begin{aligned}
\mathbf{w}^* = \text{argmin}_{\mathbf{w}} \sigma^2 \quad \text{s.t.} \quad & \mathbf{w}^T \mu = \overline{\mu} \\
& \text{other constraints} \ ,
\end{aligned}$$

where other constraints could be:

- fully invested portfolio

$$\sum_k w_k = 1 \; ,$$

- no leverage on asset k

$$w_k \le 1 \; ,$$

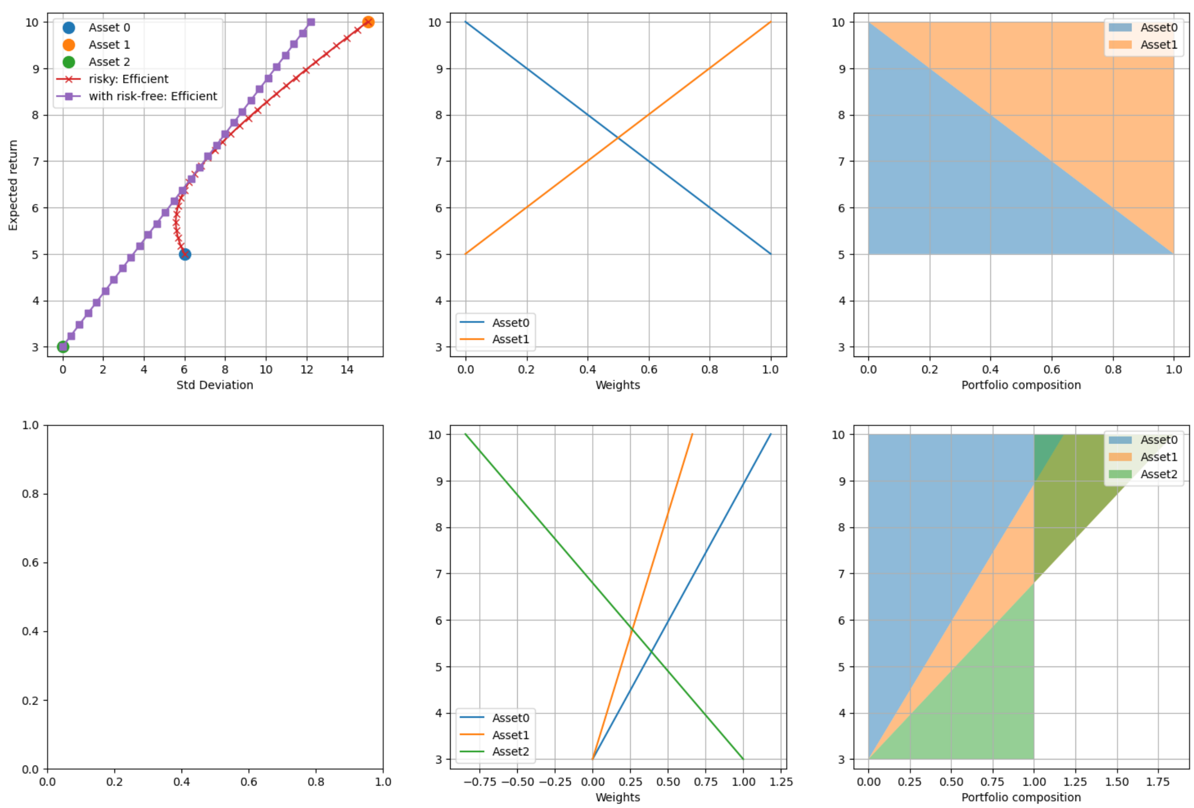
- no short-selling on asset k

$$w_k \ge 0 \; .$$

Useful arrays and functions. Functions to be used in the optimization are defined here. The optimization process aims at finding the asset allocation with minimum variance of the return, given the expected value of the return.

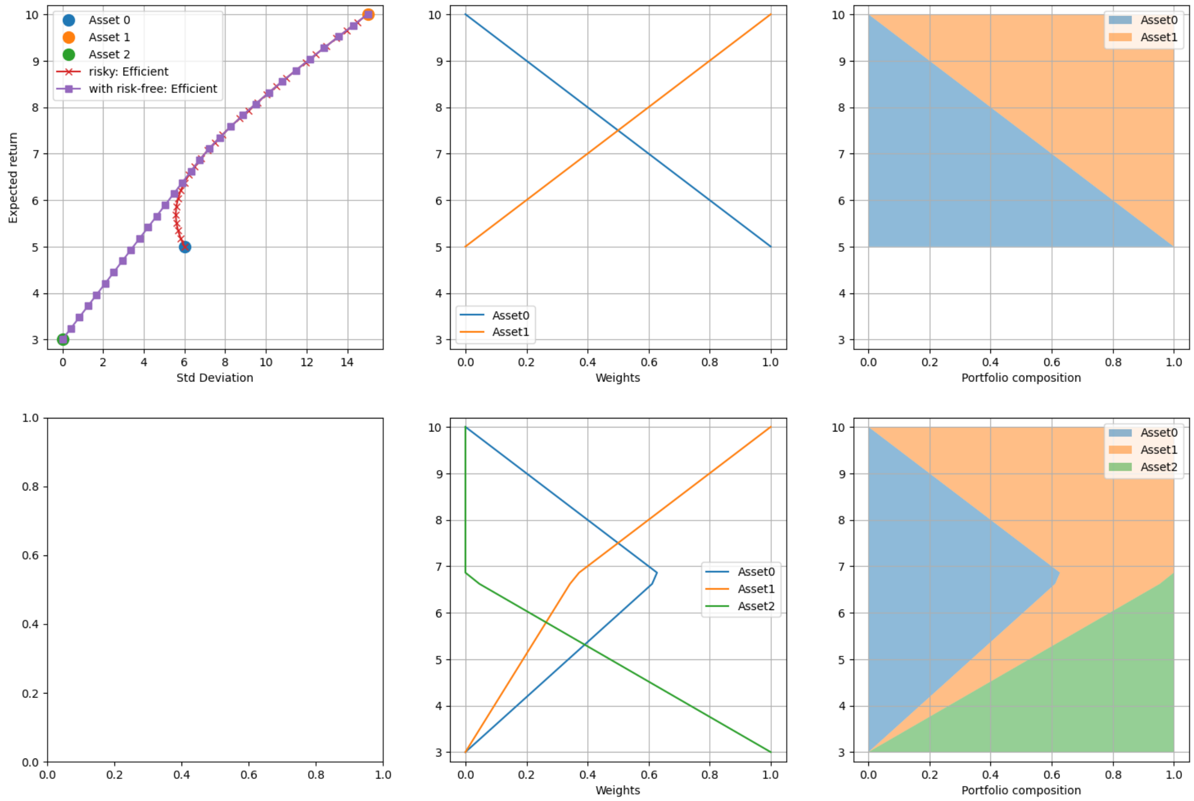
18.2 Efficient frontier and CAPM with possibility of leverage and short-selling

Plots



18.3 Efficient frontier and CAPM without leverage and short-selling

Plots.



18.4 MPT and CAPM: analytical solution

18.4.1 MPT

Analytical solution of the MPT optimization problem for a **fully invested portfolio** with **no risk-free asset**

$$\mathbf{w}^* = \text{argmin}_{\mathbf{w}} \sigma^2 \quad \text{s.t.} \quad \begin{array}{l} \mathbf{w}^T \mu = \overline{\mu} \\ \sum_{i=1}^{N} w_i = 1 \end{array}$$

without any additional constraint. Using Lagrange multiplier method, the augmented objective function

$$\widetilde{J}(\mathbf{w}; a, b) = \frac{1}{2} \mathbf{w}^T \sigma^2 \mathbf{w} - a(\mathbf{w}^T \mu - \mu) - b(\mathbf{w}^T \mathbf{1} - 1)$$

and setting its gradient equal to zero gives the following system of equations

$$\begin{cases} \mathbf{0} & = \sigma^2 \mathbf{w}^* - a\mu - b\mathbf{1} \\ 0 & = \mu^T \mathbf{w}^* - \mu \\ 0 & = \mathbf{1}^T \mathbf{w}^* - 1 \,, \end{cases}$$

or using matrix formalism

$$\begin{bmatrix} \sigma^2 & -\mu & -\mathbf{1} \\ -\mu^T & 0 & 0 \\ -\mathbf{1}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}^* \\ a \\ b \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\mu \\ -1 \end{bmatrix}$$

Without any risk-free asset, the covariance matrix is non-singular, and thus invertible. (Formally) solving the first equation for $\mathbf{w}$,

$$\mathbf{w} = \sigma^{-2}\mu a + \sigma^{-2}\mathbf{1}b \ ,$$

a system of 2 equations in 2 unknowns a, b reads

$$\begin{bmatrix} \mu^T\sigma^{-2}\mu & \mu^T\sigma^{-2}\mathbf{1} \\ \mathbf{1}^T\sigma^{-2}\mu & \mathbf{1}^T\sigma^{-2}\mathbf{1} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix} \ .$$

and thus

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} & = \frac{1}{A_{11}A_{22} - 2A_{12}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{12} & A_{11} \end{bmatrix} \begin{bmatrix} \mu \\ 1 \end{bmatrix} \\ & = \frac{1}{\Delta} \left(\begin{bmatrix} A_{22} \\ -A_{12} \end{bmatrix} \mu + \begin{bmatrix} -A_{12} \\ A_{11} \end{bmatrix} \right) \ . \end{aligned}$$

Thus, the optimal asset allocation is a 1-degree function of μ,

$$\mathbf{w}^* = \mathbf{w}_0 + \mathbf{w}_{/\mu}\mu \ .$$

and its variance is a 2-degree function of μ,

$$\begin{aligned} \sigma^{2*} & = \mathbf{w}^{*T}\sigma^2\mathbf{w}^* = \sigma_0^2 + \sigma_1^2\mu + \sigma_2^2\mu^2 = \\ & = \left(a\mu^T + b\mathbf{1}^T\right)\sigma^{-2} \underbrace{\sigma^2\,\sigma^{-2}}_{=\mathbf{I}} \left(\mu a + \mathbf{1}b\right) = \\ & = a^2\mu^T\sigma^{-2}\mu + 2ab\mathbf{1}^T\sigma^2\mu + b^2\mathbf{1}^T\sigma^{-2}\mathbf{1} = \\ & = \left(\frac{1}{\Delta}\left(A_{22}\mu - A_{12}\right)\right)^2 A_{11} + 2\frac{1}{\Delta^2}\left(A_{22}\mu - A_{12}\right)\frac{1}{\Delta}\left(-A_{12}\mu + A_{11}\right)A_{12} + \left(\frac{1}{\Delta}\left(-A_{12}\mu + A_{11}\right)\right)^2 A_{22} = \\ & = \frac{1}{\Delta^2}\left(\mu^2\left(A_{22}^2A_{11} - 2A_{22}A_{12}^2 + A_{12}^2A_{22}\right) + \mu\cdot 2\left(-A_{11}A_{22}A_{12} + A_{22}A_{11}A_{12} + A_{12}^3 - A_{11}A_{22}A_{12}\right) + \left(A_{12}^2A_{11} - 2A_{12}^2A_{11} - \right.\right. \\ & = \frac{1}{\Delta^2}\left[\mu^2A_{22}\left(A_{11}A_{22} - A_{12}\right) - 2\mu A_{12}\left(A_{11}A_{22} - A_{12}^2\right) + A_{11}\left(A_{11}A_{22} - A_{12}^2\right)\right] = \\ & = \frac{1}{\Delta}\left(\mu^2A_{22} - 2\mu A_{12} + A_{11}\right) = \\ & = \begin{bmatrix} \mu & 1 \end{bmatrix} \mathbf{A}^{-1} \begin{bmatrix} \mu \\ 1 \end{bmatrix} \ . \end{aligned}$$

As the matrix $\mathbf{A}$ is definite positive (its inverse is definite positive as well?), it follows that $\sigma^2 > 0$ for any value of μ, as expected for the value of a variance.

Some analytic geometry. The function

$$\begin{aligned} \text{Var}[r](\mu) & = \frac{1}{\Delta}\left(A_{22}\mu^2 - 2A_{12}\mu + A_{11}\right) = \\ & = \frac{A_{22}}{\Delta}\left(\mu - \frac{A_{12}}{A_{22}}\right)^2 + \frac{A_{11}}{\Delta} - \frac{A_{12}^2}{A_{22}\,\Delta} = \\ & = \frac{A_{22}}{\Delta}\left(\mu - \frac{A_{12}}{A_{22}}\right)^2 + \frac{1}{A_{22}} \ . \end{aligned}$$

is the function of a parabola, with vertex in

$$\mu_v = \frac{A_{12}}{A_{22}}$$
$$\text{Var}[r]_v = \text{Var}[r](\mu_v) = \frac{1}{\Delta}\left(\frac{-A_{12}^2 + A_{11}A_{22}}{A_{22}}\right) = \frac{1}{A_{22}} \ .$$

Using σ as an independent coordinate (and not $\text{Var}[r] = \sigma^2$)…

Properties of matrix $\mathbf{A}$

Is it positive definite? Covariance matrix is positive matrix, so for $\forall \mathbf{v}$

$$0 < \mathbf{v}\sigma^2\mathbf{v} \ ,$$

and choosing $\mathbf{v} = [\mu \quad \mathbf{1}] \begin{bmatrix} a \\ b \end{bmatrix}$, for $\forall a, b$, it immediately follows

$$\begin{aligned} 0 &< \mathbf{v}\sigma^2\mathbf{v} = \\ &= [a \quad b] \begin{bmatrix} \mu^T \\ \mathbf{1}^T \end{bmatrix} \sigma^2 \, [\mu \quad \mathbf{1}] \, [a \quad b] = \\ &= [a \quad b] \begin{bmatrix} \mu^T\sigma^2\mu & \mu^T\sigma^2\mathbf{1} \\ \mathbf{1}^T\sigma^2\mu & \mathbf{1}^T\sigma^2\mathbf{1} \end{bmatrix} [a \quad b] \ , \end{aligned}$$

and thus matrix $\mathbf{A}$ is definite positive.

18.4.2 CAPM

Analytical solution of the MPT-CAPM optimization problem, with a risk-free asset. If a risk-free asset exists, the covariance matrix is singular. However, the risk-free asset can be partitioned from the risky assets, so that the covariance matrix of the return of the risky asset is non-singular. The problem becomes

$$(\mathbf{w}, w_0)^* = \text{argmin}_{()}\sigma^2 \quad \text{s.t} \quad ...$$

…

$$\begin{bmatrix} \sigma^2 & \mathbf{0} & -\mu & -\mathbf{1} \\ \mathbf{0}^T & 0 & -\mu_0 & -1 \\ -\mu^T & -\mu_0 & 0 & 0 \\ -\mathbf{1}^T & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}^* \\ w_0^* \\ a \\ b \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ -\mu \\ -1 \end{bmatrix}$$

From the second and the fourth equation,

$$b = -\mu_0 a$$
$$w_0^* = -\mathbf{1}^T\mathbf{w}^* + 1$$

and thus

$$\begin{bmatrix} \sigma^2 & -\mu + \mu_0\mathbf{1} \\ -\mu^T + \mu_0\mathbf{1}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}^* \\ a \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -(\mu - \mu_0) \end{bmatrix}$$

whose solution reads

$$a = \frac{\mu_e}{\mu_e^T\sigma^{-2}\mu_e}$$
$$\mathbf{w}^* = \frac{\mu_e\sigma^{-2}\mu_e}{\mu_e^T\sigma^{-2}\mu_e}$$

and the relationship between the variance and the expected value of the optimal portfolios,

$$\sigma^2 = \frac{\mu_e^2}{\mu_e^T \sigma^{-2} \mu_e}$$

or the linear relation between the standard deviation of the portoflio σ and the excess return μ_e of the portfolio w.r.t. the risk-free asset,

$$\sigma = \frac{\mu_e}{\sqrt{\mu_e^T \sigma^{-2} \mu_e}} \ .$$

Solution of the linear system - details

$$\begin{aligned} \mathbf{w}^* &= \sigma^{-2} \left(\mu - \mu_0 \mathbf{1} \right) a \ , \\ \mu - \mu_0 &= \left(\mu - \mu_0 \mathbf{1} \right)^T \mathbf{w}^* = \\ &= \left(\mu - \mu_0 \mathbf{1} \right)^T \sigma^{-2} \left(\mu - \mu_0 \mathbf{1} \right) a \ , \\ a &= \frac{\mu_e}{\mu_e^T \sigma^{-2} \mu_e} \ , \end{aligned}$$

and thus

$$\mathbf{w}^* = \frac{\mu_e \sigma^{-2} \mu_e}{\mu_e^T \sigma^{-2} \mu_e} \ .$$

Eventually, the variance of the portfolio reads

$$\begin{aligned} \sigma^2 &= \mathbf{w}^{*\,T} \sigma^2 \mathbf{w} = \\ &= \mu_e^2 \frac{\mu_e^T \sigma^{-2} \sigma^2 \sigma^{-2} \mu_e}{\left(\mu_e^T \sigma^{-2} \mu_e \right)^2} = \\ &= \frac{\mu_e^2}{\mu_e^T \sigma^{-2} \mu_e} \ , \end{aligned}$$

with $\mu_e := \mu - \mu_0$ the excess desired return of the portfolio w.r.t. the risk-free asset, and $\mu_e := \mu - \mu_0 \mathbf{1}$ the vector of the excess returns of each risky asset w.r.t. the risk-free asset. Taking the square root of the last relation, a 1-degree function relates the standard deviation and the return of the portfolio,

$$\sigma = \frac{\mu_e}{\sqrt{\mu_e^T \sigma^{-2} \mu_e}} \ .$$

Tangency condition as a maximization of a measure of *risk-adjusted return*, namely **Sharpe ratio** comparing the excess return and the variance of the portfolio w.r.t. a risk-free (zero variance) asset $(\cdot)_0$ used as a benchmark $(\cdot)_b$

$$S := \frac{\mathbb{E}[r - r_b]}{\sqrt{\text{Var}[r - r_b]}} = \frac{\mathbf{w}^T \mu - \mu_0}{\sqrt{\mathbf{w}^T \sigma^2 \mathbf{w}}}$$

as the variance reads

$$\text{Var}[r - r_0] = \ldots$$

Tangency condition between optimal portfolio lines w/ and w/o risk-free asset

W/o risk-free asset

$$\sigma_r(\mu) = \sqrt{\frac{A_{22}\mu^2 - 2\mu A_{12} + A_{11}}{\Delta}}\ ,$$

with $A_{22} = \mathbf{1}^T \sigma^{-2} \mathbf{1}$, $\mathbf{A}_{11} = \mu^T \sigma^{-2} \mu$, $A_{12} = \mu^T \sigma^{-2} \mathbf{1}$, and $\Delta = A_{11}A_{22} - A_{12}$

W/ risk-free asset

$$\sigma_f(\mu) = \frac{\mu - \mu_0}{\sqrt{\left(\mu - \mu_0 \mathbf{1}\right)^T \sigma^{-2} \left(\mu - \mu_0 \mathbf{1}\right)}}\ .$$

Tangency condition

$$\sigma_r(\overline{\mu}) = \sigma_f(\overline{\mu})$$
$$\sigma_r'(\overline{\mu}) = \sigma_f'(\overline{\mu})$$

or with the variance,

…

$$0 = \mu^2 \left(\frac{A_{22}}{\Delta} - \frac{1}{\mu_e^T \sigma^{-2} \mu_e} \right) - 2\mu \left(\frac{A_{12}}{\Delta} - \frac{\mu_0}{\mu_e^T \sigma^{-2} \mu_e} \right) + \left(\frac{A_{11}}{\Delta} - \frac{\mu_0^2}{\mu_e^T \sigma^{-2} \mu_e} \right)$$

CHAPTER

NINETEEN

MERTON'S PORTFOLIO PROBLEM

Merton's portfolio problem deals with the choice of optimal fraction of investment in a risky asset π_t and the consumption c_t.

Assuming that the wealth of a family or an individual can is invested with a fraction π_t in a risky part of the portfolio with expected return μ and standard variation σ_t, and with a fraction $1 - \pi_t$ in a risk-free asset with expected return r_t and zero standard deviation, the wealth x_t of a family or an individual evolves with the SDE

$$\begin{aligned} dX_t &= r_t(1-\pi_t)X_t\,dt + \mu_t\pi_t X_t\,dt - c_t\,dt + \pi_t\sigma_t X_t\,dW_t = \\ &= r_t X_t\,dt + (\mu_t - r_t)\,\pi_t X_t\,dt - c_t\,dt + \pi_t\sigma_t X_t\,dW_t\ , \end{aligned}$$

i.e. the equation of a geometric Brownian motion with drift, the same equation that can be used to discuss *sequence risk* in investment, especially dealing with withdrawal.

Optimization problem can be solved using **continuous-time reinforcement learning**, see as an example Math:Introduction to RL, and Statistics:RL (**todo**). Optimal solution π^*_t, c^*_t is the fraction invested in the risky asset and consumption that maximises the **value function**,

$$V(x,t) = \mathbb{E}\left[\int_{s=t}^{T} e^{-\rho(s-t)}u(c_s)\,ds + e^{-\rho(T-t)}B(T)u(X_T)\middle| X_t = x\right]\ ,$$

i.e. the return of the choice defined as the cumulative discounted reward. Reward per unit time is the **utility function** $u(c_s)$, ρ is a personal discount factor that weights present and future rewards, and $B(T)$ is a bequest function.

For constant expected return and volatility of the assets, and a utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ it's possible to find an analytical solution with optimal fraction invested in the risky asset

$$\pi^* = \frac{\mu - r}{\gamma\sigma^2}\ .$$

19.1 Recursive relation and Hamilton-Jacobi-Bellman relation

Writing the value function $V(X_{t+dt}, t+dt)$ and expanding as a Taylor series up to the first order in dt, it's possible to find a **Hamilton-Jacobi-Bellman** equation from a recursive relation.

$$V(X_t + dX_t, t+dt) = \mathbb{E}\left[\int_{s=t+dt}^{T} e^{-\rho(s-t-dt)}u(c_s)\,ds + e^{-\rho(T-t-dt)}B(T)u(X_T)\middle| X_{t+dt}\right] =$$

19.1.1 Recursive relation

$$e^{-\rho t} V(X_t, t) = \mathbb{E}\left[\int_{s=t}^{T} e^{-\rho s} u(c_s)\, ds + e^{-\rho T} B(T) u(X_T) \middle| X_t \right]$$

$$\begin{aligned}
e^{-\rho(t+dt)} V(X_t + dX_t, t+dt) &= \mathbb{E}\left[\int_{s=t+dt}^{T} e^{-\rho s} u(c_s)\, ds + e^{-\rho T} B(T) u(X_T) \middle| X_{t+dt} \right] = \\
&= \mathbb{E}\left[\int_{s=t}^{T} e^{-\rho s} u(c_s)\, ds + e^{-\rho T} B(T) u(X_T) \middle| X_{t+dt} \right] - \mathbb{E}\left[\int_{s=t}^{t+dt} e^{-\rho s} u(c_s)\, ds \middle| X_{t+dt} \right] =
\end{aligned}$$

$$e^{-\rho(t+dt)} V(X_t + dX_t, t+dt) = e^{-\rho t} V(X_t, t) - e^{-\rho t} u(c_t)\, dt$$

$$d\left(e^{-\rho t} V(X_t, t)\right) = -e^{-\rho t} u(c_t)\, dt$$

$$\rho V\, dt = dV + u(c_t)\, dt$$

19.1.2 Taylor expansion

Taylor expansion of value function $v(t, x)$, valued with $x = X_t$, reads

$$\begin{aligned}
dv &= \partial_t v\, dt + \partial_x v\, dx + \frac{1}{2} \partial_{xx} v\, dx^2 + o(dt) = \\
&= \partial_t v\, dt + \partial_x v\, dX_t + \frac{1}{2} \partial_{xx} v\, dX_t^2 + o(dt) = \\
&= dt \left(\partial_t v + \partial_x v (r_t + (\mu_t - r_t)\pi_t) X_t - \partial_x v\, c_t \right) + \partial_x v\, \pi_t \sigma_t X_t\, dW_t + \frac{1}{2} \partial_{xx} v\, \left(\pi_t \sigma_t X_t\right)^2\, dt + o(dt) =
\end{aligned}$$

as $dX_t^2 = \left(\pi_t \sigma_t X_t\right)^2 dW_t^2 + o(dt) = \left(\pi_t \sigma_t X_t\right)^2 dt + o(dt)$. Taking expected value,

$$dV = dt\, \mathbb{E}\left[\partial_t v + \partial_x v (r_t + (\mu_t - r_t)\pi_t) X_t - \partial_x v\, c_t + \frac{1}{2} \partial_{xx} v\, \left(\pi_t \sigma_t X_t\right)^2 \right] .$$

19.1.3 Recursive relation for optimal solution

First order in dt reads

$$\rho V^* = \max_{\pi_t, c_t} \left\{ \partial_t V^* + \partial_x V^* \left[(\pi_t(\mu - r) + r) X_t - c_t \right] + \frac{1}{2} \partial_{xx} V^* (\pi \sigma_t X_t)^2 + u(c_t) \right\} = \max_{\pi_t, c_t} \Phi$$

19.1.4 Optimality condition

Zero gradient is a necessary condition for local extreme points,

$$\begin{cases}
0 = \partial_{\pi_t} \Phi(\pi_t^*, c_t^*) = (\mu - r) X_t \partial_x V^* + \partial_{xx} V^* \pi_t^* \left(\sigma_t X_t\right)^2 \\
0 = \partial_{c_t} \Phi(\pi_t^*, c_t^*) = -\partial_x V^* + \partial_{c_t} u(c_t^*)
\end{cases}$$

and thus

$$\pi_t^* = -\frac{\partial_x V^*}{\partial_{xx} V^*} \frac{\mu - r}{X_t \sigma^2}$$

Necessary conditions on $u()$ for the Hessian to be definite negative

todo

19.1.5 Example: utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$

$$\begin{cases} \pi_t^* = -\dfrac{\partial_x V^*}{\partial_{xx} V^*} \dfrac{\mu - r}{X_t \sigma^2} \\ c_t^* = (\partial_x V^*)^{-\frac{1}{\gamma}} \end{cases}$$

19.1.6 Example: value function $V^*(t,x) = f^\gamma(t) \frac{x^{1-\gamma}}{1-\gamma}$

With value function $V^*(x,t) =$ ... and bequest function $B(T) = \varepsilon^T$

$$\dot{f}(t) = \nu\, f(t) - 1$$

with final condition $f(T) = \varepsilon$, and

$$\nu = \frac{1}{\gamma} \left\{ \rho - (1-\gamma) \left(\frac{(\mu - r)^2}{2\sigma^2 \gamma} + r \right) \right\} .$$

The solution of the ODE reads

$$f(t) = \begin{cases} \dfrac{1}{\nu} + \left(\varepsilon - \dfrac{1}{\nu} \right) e^{-\nu(T-t)} & , \quad \text{if } \nu \neq 0 \\ T - t + \varepsilon & , \quad \text{if } \nu = 0 \end{cases}$$

Thus

$$\pi_t^* = \frac{\mu - r}{\gamma \sigma^2}$$
$$c_t^* = \frac{X_t}{f(t)}$$

CHAPTER

TWENTY

REBALANCING

In this Notwbook, two strategies on a 2-asset portfolio are discussed and compared:

- **rebalanced portfolio**, after each period
- **buy-and-hold portfolio**, without rebalancing

Effects of rebalancing and conditions for **rebalancing premium** are discussed: sometimes the expected log-return of the rebalanced portfolio may exceed the expected log-return of each single asset.

20.1 Libraries, parameters and useful functions

Libraries are imported and useful functions to treat conics below are defined here

20.1.1 Libraries

20.1.2 Parameters

20.1.3 Functions for conics

20.2 Rebalanced portfolio

Let the **1-period return** of the assets be normal (**todo** *is this necessary? Can't one rely on central limit theorem? How long the summation must be for convergence to normal distribution, in presence of* ***heavy-tails*** *distribution? If one can't rely on central limit theorem, let use numerical methods to investigate the effect of heavy tails distributions*),

$$\mathbf{r} \sim \mathcal{N}\left(\mu, \sigma^2\right) \ .$$

Compound return of the portfolio has **expected value**

$$\mu_p^c = \mathbb{E}[r_p^c] = \mathbf{w}^T \mu - \frac{1}{2} \mathbf{w}^T \sigma^2 \, \mathbf{w}$$

and **variance**

$$\sigma^2_{r_p^c} = \mathbb{E}\left[(r_p^c - \mu_p^c)^2\right] = \dots = \mathbf{w}^T \sigma \mathbf{w}$$

20.2.1 Shannon demon

Sometimes the expected value of the compound return of the rebalanced portfolio can be larger than the expected return of each asset class.

$$\mu_k^c = \mathbb{E}[r_k^c] = \mu_k - \frac{\sigma_k^2}{2}$$

Example: 2-asset portfolio. As an example, the expected value of the compund return of a 2-asset rebalanced portfolio,

$$\mathbb{E}[r_p^c] = w_1\mu_1 + w_2\mu_2 - \frac{1}{2}(w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1\sigma_2 + w_2^2\sigma_2^2)$$

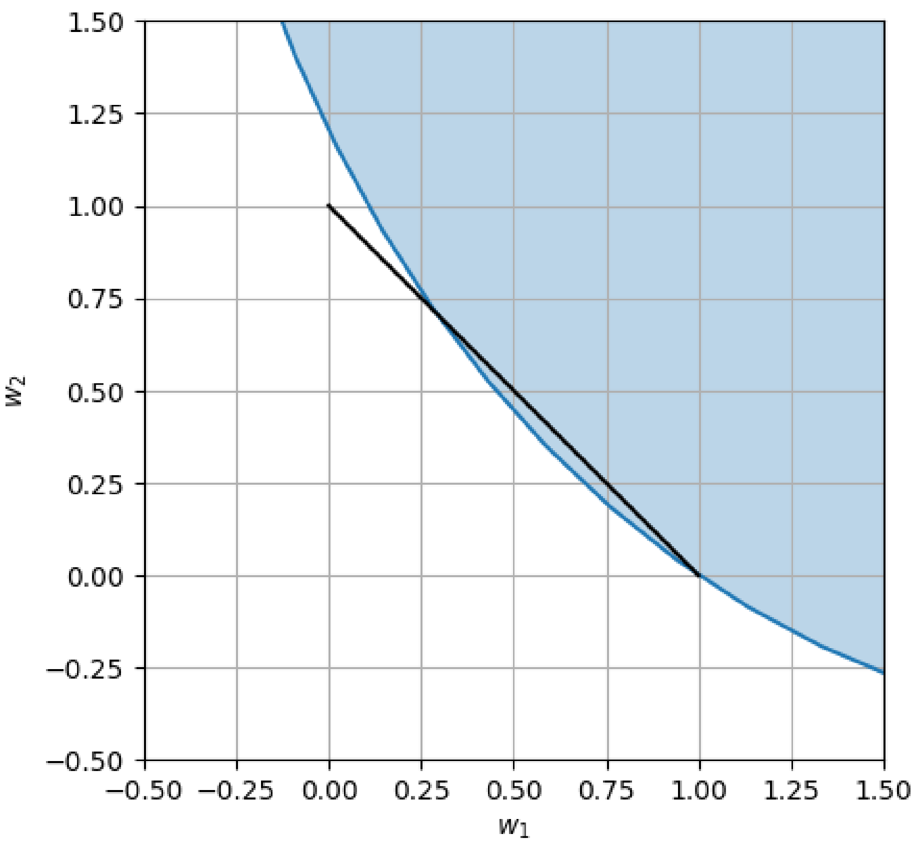
Using w_1, w_2 as independent variables, for any value of the expected return, the expression of the return itself can be represented as a **conic section** in the w_1, w_2-plane. In particular,

- for $\rho \neq 1$, it's an **ellipse** ($\Delta = B^2 - 4AC < 0$),
- for $\rho = 1$, it's a **parabola** ($\Delta = 0$)

Portfolio allocation. Some constraints may hold on portfolio allocation:

- fully-invested: $w_1 + w_2 = 1$
- no short-selling: $w_1,\ w_2 \geq 0$
- no leverage: $w_1,\ w_2 \leq 1$

```
interactive(children=(FloatSlider(value=-0.25, description='rho', max=1.0, min=-1.
↪0, step=0.01), FloatSlider(v…
```



This plot represents in **blue** asset allocations of the balanced 2-asset portfolio with expected value of the compound return larger than the compound return of any individual asset. **Black line** represents all the possible allocations of a fully invested portfolio, $w_1 + w_2 = 1$ with no leaverage $w_k \leq 1$ and not short selling $w_k \geq 0$.

20.3 Comparison of portfolios: realizations of stochastic processes

In this section, rebalanced portfolio and buy-and-hold portfolio are compared. Different realizations of these two portfolio strategies are built, and used to build statistics, and discuss their properties in terms of **compound return**, **drawdowns**,…

Note. Here, 1-period returns are modelled as *normal random variable* so far. Anyways, it's possible (and suggested) to implement the most suited random process for modelling the return of the desired assets.

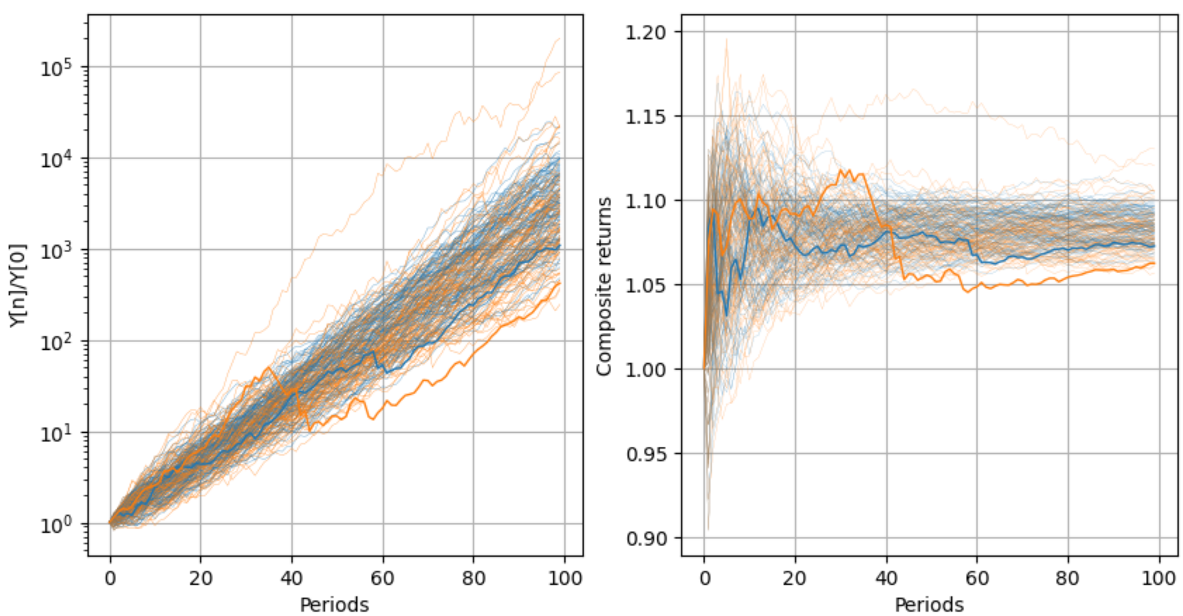
20.3.1 Useful functions

A useful function is introduced here to build correlated random variables with the desired expected values and covariance.

Main Colab notebook can be found here: https://colab.research.google.com/drive/1n5py0Zf8i3_jrTTk0AR7Noq2kBYwpaqx?authuser=1#scrollTo=gmbjbprjCHto

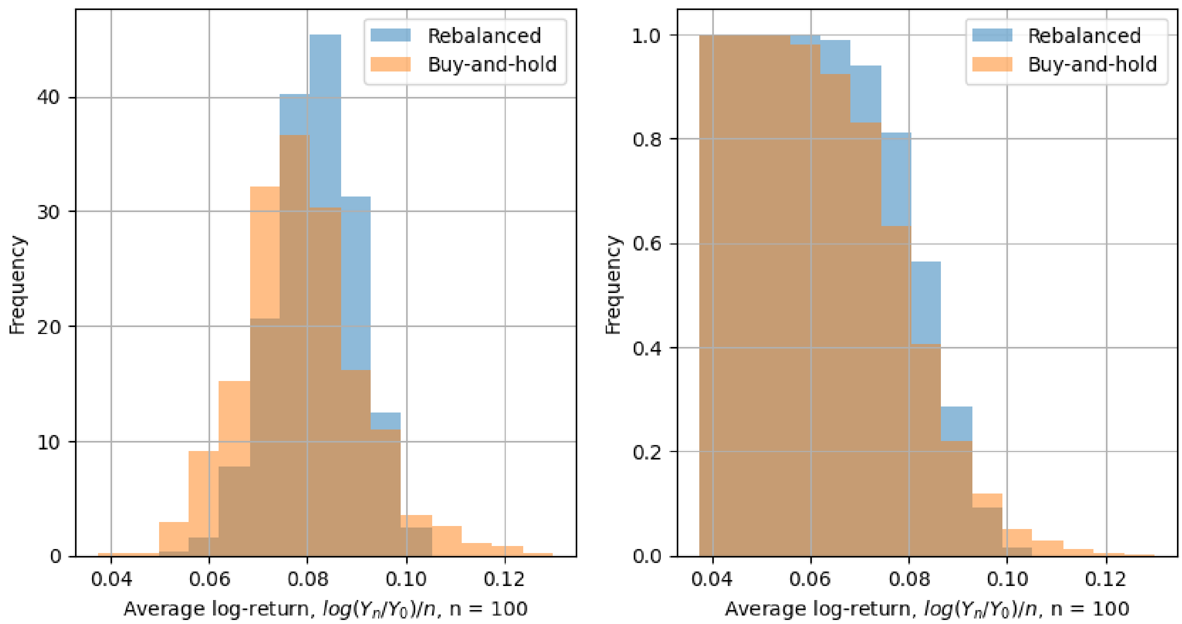
20.3.2 Realizations

```
[<matplotlib.lines.Line2D at 0x7fd10171efa0>]
```



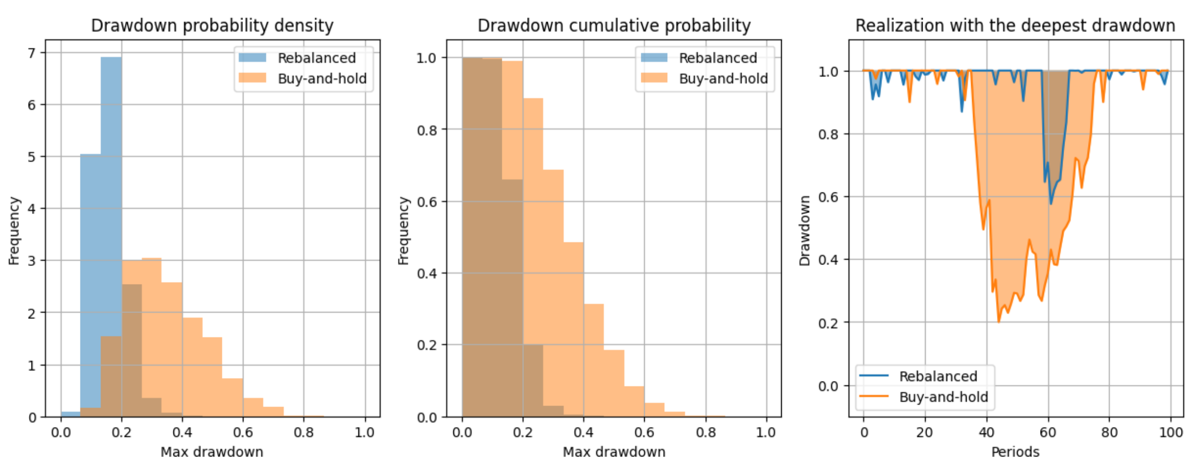
20.3.3 Composite return

```
Rebalanced portfolio. Compsite return
 Exp. value: 0.085
 Std. dev. : 0.009
Buy-and-Hold portfolio. Compsite return
 Exp. value: 0.082
 Std. dev. : 0.013
```



20.3.4 Maximum drawdown

```
Text(0.5, 1.0, 'Realization with the deepest drawdown')
```



Part VI

Historical Data

CHAPTER

TWENTYONE

HISTORICAL DATA

CHAPTER

TWENTYTWO

USA EQUITY MARKET

In 1926 the Stardard Statistics Company developed a 90-stock index. On March 4, 1957, Standard and Poor's - the name of the company after the merging of the SSC and the Poor's Publishing - expanded the index to the current number of 500 companies, renamed the S&P500 Stock Composite Index.

On August 31, 1976, The Vanguard Group launched the first mutual fund to retail investors that tracked the index.

Table of contents.

- **Return** over different time frames, both in nominal and real value
- **Sectors**

22.1 Return

- Time history of the total return index
- Return over different time spans: probability function and statistics, composition of returns
- Volatility: drag, volatility clusters,…

22.1.1 Total Return - Shiller data

Shiller data, from 1871 to 2025

```
count    1854.000000
mean        0.006515
std         0.040827
min        -0.261879
25%        -0.013347
50%         0.009404
75%         0.029377
max         0.524294
Name: return_month, dtype: float64
Skewness: 0.5564109460860193
Kurtosis: 17.948214818518505
```

22.2 Sectors

Message: market composition changes - rise and fall of dominant sectors; diversification can disappear as market concentration in one or few sectors increases; survivorship and return bias.

Methods:

- Evolution of largest companies of a time range over longer time ranges
- SPDR sector ETFs, available since 1998/12

References:

- Wikipedia, top 10 companies 2000-2025, https://en.wikipedia.org/wiki/List_of_public_corporations_by_market_capitalization
- Forbes, top 50 companies: 1917, 1967, 2017, https://www.forbes.com/sites/jeffkauflin/2017/09/19/americas-top-50-companies-1917-2017/
- finhacker, top 20 US companies 1989-2026, https://www.finhacker.cz/en/top-20-sp-500-companies-by-market-cap/
- finaeon, 200 years of market concentration, https://finaeon.com/200-years-of-market-concentration/
- JPM Asset Management, picture in `media` folder

22.3 References

- Kenneth Franch Data Library: industry portfolios by market cap, long history, clean and ideal for education, https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
- Robert Shiller data
  - http://www.econ.yale.edu/~shiller/data.htm
  - https://shillerdata.com/
- Wikipedia for top companies: scratch largest companies by mkt cap every 5-10 year and then show the evolution of the market cap

Part VII

Extra

CHAPTER

TWENTYTHREE

EXTRA AND RANDOM

CHAPTER

TWENTYFOUR

EURISTHICS AND HISTORICAL CORRELATIONS

24.1 Expected returns in the stoch market

24.1.1 Shiller P/E and S&P500 10-year annualised forward returns

- from Invesco, Applied philosophy: The Shiller P/E and the S&P500 returns

24.1.2 Bogle Expected Return Formula

- Comment by *Ben Carlson*, on his website

Part VIII

People

CHAPTER

TWENTYFIVE

RESOURCES, PEOPLE AND FIRMS

25.1 Tools

These tools have not been tested and verified here. I decline any responsibility about their use.

25.1.1 Simple tools for investors

[Simple tools for investors] (https://www.simpletoolsforinvestors.eu/index.shtml): **bond** monitor, calculators, tools (minus-eater, laddering,...)

25.1.2 Interactive Asset Allocation - by Research Affiliates

Expected return vs. volatility plot of different asset classes on different time horizons by *Research Affiliates*.

25.1.3 Gregory Gundersen Blog

Gregory Gundersen blog

25.2 People

25.2.1 Arnott, Robert

Founder of *Research Affiliates*.

25.2.2 Faber, Meb(ane)

Co-founder and Chief Investment Officier at *Cambria Investment Management*.

25.2.3 Wigglesworth, Robin

Financial Times' global finance correspondent, author of Trillions a book on the past, present and future of passive investing.

25.2.4 Zweig, Jason

Columnist for the Wall Street Journal since 2008. A Safe Heaven for Intelligent Investor.

25.2.5 Ritholtz, Barry

25.2.6 Maggiulli, Nick

25.2.7 Carlson, Ben

Director of Institutional Asset Management at *Ritholtz Wealth Management*. Author of the webiste A Wealth of Common Sense

25.2.8 Green, Micheal

Forseen **"Volmageddon"** of the 5 February 2018, the collapse of invesrse ETFs or ETP: VIX, SVXY and VMIN, as a consequence of the VIC daily surge from 17 to 37 (approximately +115%, that erased daily inverse products)

Even a moderate (~4%) equity sell-off can unleash devastating volatility swings if the short-vol market is crowded. Rebalancing demands can worsen volatility when liquidity dries up near market close.

- *The Bull*, 192. Micheal Green: perché l'investimento passivo distorce il mercato (e come comportarci)

25.2.9 Yardeni, Edward

25.3 Firms

25.3.1 Reserach Affiliates

Founded by *Robert Arnott*

25.3.2 Cambria Investment Management

Co-founded by *Mebane Faber*

25.3.3 Ritholtz Wealth Management

Barry Ritholtz, *Nick Maggiulli*, *Ben Carlson*

CHAPTER TWENTYSIX

THE BULL

"The Bull, il tuo podcast di finanza personale", Riccardo Spada. Youtube channel

- Jason Zweig, American financial journalist, columnist for the WSJ since 2008. Jason Zweig - A Safef Haven for Intelligent Investor
  - **224** what it takes to become an intelligent investor
- Nick Maggiulli, COO for Ritholtz Wealth Management LCC, financial blogger at Of Dollars And Data - Act Smarter. Live Richer. Author of "Just Keep Buying" about the power of compounding.
  - **221** Just keep buying
- Davide Serra, founder and CEO of Algebris Investment
  - **216** The change we're living and consequences for investors
- William Bernstein, american financial theorist and neurologist
  - **214** The 4 pillars of investing and how to manage risk and uncertainty
- Barry Ritholtz
  - **206** How **not** to invest
- Robin Wigglesworth
  - **203** the case for passive investing: navigative crises through simplicity
- Ben Carlson, Director of Institutional Asset Management at *Ritholtz Wealth Management*. Author of the webiste A Wealth of Common Sense
  - **200** Investing is mostly a matter of common sense (and patience)
- Robert Arnott
  - **196** Fundamental investing
- Micheal Green
  - **192** Why passive investing is distorting the markets
  - Haddad, Valentin, associate professor at the UCLA: how competitive is stock market? why is asset demand inelastic?
- Edward Yardeni,
  - **183** Why should we be optimistic?
- Meb Faber,
  - **180** Investing with Common Sense
- Eugene Fama, efficient market, **factor** investing, and factor ETFs

- **164**

PROOF INDEX